

Weakly disordered spin ladders

E. Orignac and T. Giamarchi

Laboratoire de Physique des Solides, Université Paris-Sud, Bât. 510, 91405 Orsay, France¹

Abstract

We analyze an anisotropic spin 1/2 two legs ladder in the presence of various type of random perturbations. The generic phase diagram for the pure system, in a way similar to spin one chains, consists of four phases: an Antiferromagnet, a Haldane gap or a Singlet phase (depending on the sign of interchain coupling) and two XY phases designated by XY1 and XY2. The effects of disorder depend crucially on whether it preserves XY symmetry (random field along z and random exchange) or not (random anisotropy or random XY fields). In all cases we computed the new phase diagram and the correlation length for the disordered system. The ladder exhibits a remarkable stability to disorder with XY symmetry. Not only the singlet phases but also the **massless** XY1 phase are totally unaffected by disorder in stark contrast with the single chain case. Even in the presence of an external magnetic field breaking the spin gap most of the phase diagram (including the XY point) remains unaffected by the disorder, again in opposition with the single chain case. This stabilization towards disorder is similar, albeit stronger, to the one occurring for fermionic ladders. On the other hand the XY2 phase is very strongly suppressed. Disorder breaking XY symmetry has much stronger effects and most of the phases at the exception of the singlet one are now destroyed. Interestingly, the disordered XY1 phase has a much shorter correlation length than the disordered XY2 phase contrarily with the case of perturbations preserving the rotation symmetry around the z axis. The ladder system is thus in fact much more anisotropic than its single chain counterpart with identical exchanges. Finally we examine the case of strong disorder or weak interchain exchange for XY symmetric disorder. Close to the isotropic point, when the interchain exchange is increased the system undergoes a transition between two decoupled disordered spin 1/2 chains to a singlet ladder unaffected by disorder. For more XY like systems, the transition occurs in two steps with a **non disordered** XY1 phase occurring between the decoupled chains and the singlet phase. Comparison with other disordered gapped systems such as spin one chains is discussed.

I. INTRODUCTION

One dimensional antiferromagnetic spin 1/2 systems are fascinating systems. They can present quasi-long range order, without symmetry breaking yet with critical spin spin correlation functions and diverging magnetic susceptibilities². Not surprisingly, owing to the Jordan-Wigner transformation, such properties are remarkably similar to the ones of interacting one dimensional fermions and are known under the generic name of Luttinger liquid physics^{3,4}. Experimental realization of spin 1/2 chain provide a good confirmation of these remarkable properties^{5,6}. In addition, in a way reminiscent of one dimensional fermions, spin chains are extremely sensitive to disorder. In fact using the Jordan-Wigner transformation⁷, spin chains can be mapped onto one dimensional spinless fermions, where the J_z coupling gives rise to an interaction term. Spin 1/2 systems are thus just another, and maybe more convenient experimental realization of disordered interacting fermions (or bosons)^{8,9}. The transition to the disordered phase has been analyzed numerically in Refs. 10–12 in good agreement with renormalization group treatments^{8,9}.

Recently it was realized that these remarkable properties get drastically modified when spin 1/2 chains are coupled together leading to the so called ladder system¹³. In this case, in a way very similar to the Haldane spin-S problem^{14,15}, a gap is found to open for an even number of chains while the system remains massless for an odd number of chains. This phenomenon has been thoroughly investigated both analytically^{16–18} and numerically^{19–23}. In the recent years, progress in solid-state chemistry resulted in the apparition of compounds with a 2-legs ladder structure such as SrCu_2O_3 , $(\text{La,Sr,Ca})_{14}\text{Cu}_{24}\text{O}_{41}$ and $\text{Cu}_2(1,4\text{-Diazacycloheptane})_2\text{Cl}_4$, in which the theoretical predictions about the gap have received an eclatant confirmation^{24–28}.

Given the new physics present in these systems it is of paramount importance both from a theoretical point of view and for practical experimental systems, to investigate the effects of disorder in these systems. This is especially true for the two leg ladder system for which zinc doping experiments are already performed²⁹. Due to the formation of a gapped singlet phase for the isotropic system it is somehow natural, although not always trivial, to expect that weak disorder will have no effect. Most of the studies about the influence of disorder have been thus devoted so far to the effects of strong disorder on such a gapped phase either in ladders or in other gapped systems^{30–34}. However, in the spin 1 chain system, the gap is proportional to the intra chain exchange, whereas in the ladder system, the gap is directly controlled by the interchain hopping J_\perp , and thus can be made small at will compared to the intra chain exchange J permitting the use of perturbative methods to analyze the competition of randomness and singlet formation. Moreover, very little is known on the effects of disorder on the anisotropic system, and in particular in its gapless phases. A surprising effect found for fermionic two leg ladders^{35,36} in which a gap opens in all the modes except the symmetric charge mode^{37,38} is a remarkable stability of the ladder system to disorder. Whether a similar stabilization by gap opening exists for two coupled spin chains needs to be investigated. We undertake such a study in the present paper. Quite remarkably we find that an anisotropic spin ladder is completely stable **even** in the gapless phase any weak disorder respecting the XY symmetry. This is in stark contrast with a single chain or even with ladders of spinless fermions.

The plan of the paper is the following: In section II, we recall the basics of the bosoniza-

tion treatment of spin chain systems. In order to study the effects of disorder on the anisotropic spin ladder we first investigate in details in Section III the phase diagram of the pure system. In addition to the three phases discussed in Ref. 16, namely a Singlet phase, an XY1 phase and the antiferromagnetic phase, we find another XY phase, the XY2 phase, in analogy with the case of spin one chains¹⁵. We discuss the possibility of observing this XY2 phase at intermediate coupling. In section IV we investigate the response of the two chain system with respect to weak disorder preserving the rotational symmetry around the z axis, i.e. random fields parallel to the z axis and random exchange. The spin ladder shows a remarkable stability to these perturbations even in the gapless phase. We also consider the disordered ladder under a magnetic field, able to suppress the gapped phase. Finally we study the crossover between a single chain and the ladder behavior obtained by increasing the interchain coupling. Given the remarkable stability of the ladder to random perturbations respecting the XY symmetry it is interesting to also investigate that breaks rotational symmetry around the z axis. We thus study in section V random anisotropy and random fields in the plane perpendicular to the z axis. Quite surprisingly the effect of these perturbations is opposite to the one respecting the XY symmetry: the XY1 gets very unstable, whereas the XY2 phase is much less affected. Finally conclusions and open questions can be found in Section VI.

II. BOSONIZATION OF ONE DIMENSIONAL SPIN 1/2 CHAINS

In this section, the bosonization of spin 1/2 chains is briefly recalled. For the sake of definiteness, an XXZ spin chain will be considered. It is described by the following Hamiltonian:

$$H_{\text{XXZ}} = \frac{J_{xy}}{2} \sum_i (S_{i+1}^+ S_i^- + S_i^+ S_{i+1}^-) + J_z \sum_i S_{i+1}^z S_i^z \quad (1)$$

Spin chains Hamiltonians can be transformed into interacting 1 dimensional fermion systems by expressing the spin operators S^+ , S^- , S^z in terms of fermion operators a^\dagger, a using the Jordan Wigner transformation^{2,7,39}

$$\begin{aligned} S_i^+ &= (-)^i a_i^\dagger \cos \left(\pi \sum_{j=0}^{i-1} a_j^\dagger a_j \right) \\ S_i^- &= (-)^i \cos \left(\pi \sum_{j=0}^{i-1} a_j^\dagger a_j \right) a_i \\ S_i^z &= a_i^\dagger a_i - \frac{1}{2} \end{aligned} \quad (2)$$

This changes the XXZ model into a model of spinless fermions with nearest neighbor interaction described by the Hamiltonian

$$H = -J_X/2 \sum_n (a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n) + J_Z \sum_n (a_i^\dagger a_i - \frac{1}{2})(a_{i+1}^\dagger a_{i+1} - \frac{1}{2}) \quad (3)$$

$J_z = 0$ corresponds to free fermions and the zero magnetization sector corresponds to an half-filled band with Fermi-wavevector $k_F = \frac{\pi}{2a}$, a being the lattice spacing.

To treat the problem with a finite J_z , it is convenient to use the boson representation of one-dimensional fermion operators. Only the final results are given here since details about the method can be found in various places in the literature^{2-4,40}. A detailed derivation with the same notations can be found in the appendix A of Ref. 36. The spin operators are given by

$$\begin{aligned} S^+(x) &= \frac{e^{-i\theta(x)}}{\sqrt{2\pi a}} \left[e^{-i\frac{\pi x}{a}} + \cos 2\phi(x) \right] \\ S_z(x) &= -\frac{1}{\pi} \partial_x \phi + e^{i\frac{\pi x}{a}} \frac{\cos 2\phi(x)}{\pi a} \end{aligned} \quad (4)$$

Where $S^+(x) = \frac{S_n^+}{\sqrt{a}}$, $S^z(x) = \frac{S_n^z}{a}$ for $x = na$, a being the distance between two nearest neighbors sites along the chain. The phase ϕ is related to the average density of fermions (or equivalently to the uniform spin density along z) by $S_z(x) = -\frac{1}{\pi} \partial_x \phi$, whereas θ is connected to the conjugate momentum Π of ϕ (such that $[\phi(x), \Pi(x')] = i\delta(x - x')$) by $\theta(x) = \int_{-\infty}^x dy \Pi(y)$. In a very crude sense ϕ, θ can be viewed as the polar angles of a spin. The Hamiltonian (1) gives

$$H_{\text{XXZ}} = \int \frac{dx}{2\pi} \left[uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x \phi)^2 \right] + \frac{2\Delta}{(2\pi a)^2} \int dx \cos(4\phi) \quad (5)$$

This is a sine-Gordon Hamiltonian with $\beta = 4$. The cosine term comes from the so called Umklapp process. Such processes are possible as $4k_F$ is equal to a reciprocal lattice vector^{2,39-41}. The coupling constant can be determined in perturbation in the interaction J_z and gives for $J_z \ll J_{xy}$

$$\begin{aligned} K &= \left(1 + \frac{4J_z}{\pi J_{xy}} \right)^{-1/2} \\ u &= aJ_X \left(1 + \frac{4J_z}{\pi J_{xy}} \right)^{1/2} \\ \Delta &= -J_z a \end{aligned} \quad (6)$$

The range of validity of (5) is much larger than the simple perturbative regime provided the correct constant K and u are used^{41,42}. The isotropic point $J_z = J_{xy}$ (Heisenberg model) corresponds in the bosonization description to $K^* = \frac{1}{2}, \Delta^* = 0$ as can be seen from the scaling dimensions of S^+ and S^z (see Eqs. (A4) and (A3) of appendix A). These correlation function become identical, as they should for the Heisenberg point, only for $K = \frac{1}{2}$. At that point $\cos(4\phi)$ is marginally irrelevant. For larger values of J_z ($J_z > J_{xy}$) the cosine term becomes relevant. A gap opens in the excitation spectrum of the ϕ field, leading to an insulating charge density wave state for the fermions and equivalently to an Ising order along z for the spin chain (see e.g. Ref. 39-42).

Another instability of the XXZ model can be guessed from (6). For $J_z \rightarrow -\frac{\pi}{4}J_{xy}$, $K \rightarrow \infty$. This is the sign of a transition to a fully polarized ferromagnetic state. The

exact solution shows that the ferromagnetic transition takes place for $J_z = -J_{xy}$ ³⁹. In the following, ferromagnetic systems will not be considered as their behavior is very similar to the one of classical systems at $T = 0K$.

III. TWO COUPLED SPIN 1/2 CHAINS

In this section, two XXZ chains coupled by an exchange term are considered using bosonization techniques. The exchange coupling is of the form :

$$H_{\text{interchain}} = \sum_{i,\alpha=x,y,z} J_{\perp}^{\alpha} S_{i,1}^{\alpha} S_{i,2}^{\alpha} \quad (7)$$

where for simplicity, as for the intrachain Hamiltonian, planar isotropy $J_{\perp}^x = J_{\perp}^y = J_{\perp}^{x,y}$ is assumed.

The total Hamiltonian is : $H_{\text{pure}} = H_{\text{XXZ},1} + H_{\text{XXZ},2} + H_{\text{interchain}}$. Each single chain Hamiltonian can be expressed in terms of fermions operators using the Jordan-Wigner transformation like in section II. One has to take care that the “fermions” operators should have anticommutation relations inside one chain but *commutation* relations between the chains to respect the spin commutation relations. This causes no modification of the bosonized expressions (4) for the spin operators. Indeed it is easy to check that (4) preserves all the correct spin commutation relations. H_{XXZ} has been obtained in section II and only the bosonized expression of $H_{\text{interchain}}$ is needed. Using (4) and keeping only the most relevant operators one obtains^{15,16,43}:

$$H_{\text{interchain}} = \int \left[\frac{2g_1}{(2\pi a)^2} \cos(\theta_1 - \theta_2) + \frac{2g_2}{(2\pi a)^2} \cos 2(\phi_1 - \phi_2) + \frac{2g_3}{(2\pi a)^2} \cos 2(\phi_1 + \phi_2) \right] dx \\ + J_{\perp}^z a \int dx \frac{\partial_x \phi_1 \partial_x \phi_2}{\pi^2} \quad (8)$$

Where :

$$g_1 = \pi J_{\perp}^{x,y} a \\ g_2 = J_{\perp}^z a \\ g_3 = J_{\perp}^z a \quad (9)$$

The total Hamiltonian H_{pure} is rewritten in terms of the fields $\phi_a = \frac{\phi_1 - \phi_2}{\sqrt{2}}$ and $\phi_s = \frac{\phi_1 + \phi_2}{\sqrt{2}}$ giving

$$H_{\text{pure}} = H_s + H_a \\ H_s = \int \frac{dx}{2\pi} \left[u_s K_s (\pi \Pi_s)^2 + \frac{u_s}{K_s} (\partial_x \phi_s)^2 \right] + \frac{2g_2}{(2\pi \alpha)^2} \int dx \cos(\sqrt{8} \phi_s) \\ H_a = \int \frac{dx}{2\pi} \left[u_a K_a (\pi \Pi_a)^2 + \frac{u_a}{K_a} (\partial_x \phi_a)^2 \right] + \frac{2g_3}{(2\pi \alpha)^2} \int dx \cos(\sqrt{8} \phi_a) + \frac{2g_1}{(2\pi \alpha)^2} \int dx \cos(\sqrt{2} \theta_a) \quad (10)$$

Where :

$$\begin{aligned}
u_s &= u \left(1 + \frac{K J_\perp^z a}{2\pi u} \right), & K_s &= K \left(1 - \frac{K J_\perp^z a}{2\pi u} \right) \\
u_a &= u \left(1 - \frac{K J_\perp^z a}{2\pi u} \right), & K_a &= K \left(1 + \frac{K J_\perp^z a}{2\pi u} \right)
\end{aligned} \tag{11}$$

u, K being given by (6) for a XXZ model and small J_z , and more complicated expressions in the general case^{41,42}. If the correct values of u and K are used (10) is valid irrespective of the values of J_z and J_{xy} and only assumes $J_\perp a \ll \frac{u}{K}$. Because of the choice of an hermitian string operator, the $\cos(4\phi_1), \cos(4\phi_2)$ terms in $H_{\text{XXZ},1}$ and $H_{\text{XXZ},2}$ respectively can be dropped, since they always give terms that are less relevant than the terms that come from $H_{\text{interchain}}$. Note that the interchain coupling in (10) is different from the one existing in spinless fermions ladders^{37,35,36}. This is due to the presence of the string operator (2) in the fermion-spin equivalence. Therefore two coupled spin chains have a different physics that cannot be directly deduced from the fermionic results^{35,36}. This distinction between spin chains and spinless fermions will be crucial in the presence of disorder.

Eqs. (8) and (10) have been derived, for particular cases or for physically related systems, in various places in the literature. Such Hamiltonians were first introduced to describe a quantum spin-one chain^{15,43}. For this problem, although the form of the Hamiltonian are identical g_1, g_2, g_3 are related to parameters with a physical meaning different of the one of J_\perp . For two coupled spin chains (8) and (10) have been obtained and investigated^{16,44} for the particular case of isotropic interchain coupling ($J_\perp^{x,y} = J_\perp^z$). This corresponds to taking $g_2 = g_3 = g_1/\pi$ in (10). More recently, another derivation for isotropically coupled spin chains has been obtained by a different method¹⁸. Instead of using a Jordan-Wigner transformation, Ref. 18 uses a representation in term of localized fermion. Such a representation is particularly well adapted to describe an isotropic coupling since the superexchange leading to the spin Hamiltonian is automatically generated. Interestingly one recovers the same bosonized Hamiltonian than (10) (with $K = 1/2$ due to isotropy), but with a different relation between the coefficients g . In Ref. 18 the coefficient satisfy $g_2 = g_3 = g_1 \propto J_\perp$. This difference comes from the fact that in the present derivation the calculation of the coefficients g is perturbative and the expression (9) is accurate for $J_z \ll J_{xy}$. In going to the isotropic point these coefficients get renormalized by irrelevant terms that do not affect the form of the low energy Hamiltonian (10), but can change the explicit value of the coefficients compared to (9). Comparison of the two limits $J_z \ll J_{xy}$ and $J_z \sim J_{xy}$ shows that this renormalization effects are rather weak, and that (10-11) provide an extremely accurate description of the physics of the ladder system for all values of J_z , provided one does not need the *precise* values of the coefficients g .

Using (10) the phase diagram of the pure system can be derived. Since the main goal of this paper is to analyze the additional effects of disorder, we restrict ourselves for the study of the pure system to isotropic interchain coupling but the intrachain coupling is arbitrary. Nevertheless, (10) can be used in more general cases. The derivation is technically similar to the one for the spin one chain¹⁵, but gives of course different physical phases. Since H_s is a standard sine-Gordon Hamiltonian (see appendix A) its spectrum has a gap if $K_s < 1$ and is gapless otherwise. The treatment of H_a is a bit more subtle. $\cos(\sqrt{8}\phi_a)$ and $\cos(\sqrt{2}\theta_a)$ have respective scaling dimension of $2K_a$ from (A3) and $(2K_a)^{-1}$ from (A4) so that both are relevant for $1/4 < K_a < 1$. However, the coefficients of these operators are of the same

order of magnitude. Therefore, the most relevant operator is the first to attain the strong coupling regime under renormalization. In the strong coupling regime this operator takes a mean value that minimizes the ground state energy. θ_a and ϕ_a being conjugated, when one of them develops a mean value the exponentials of the other one have zero expectation values and exponentially decaying correlations (see appendix A). Therefore, ϕ_a acquires a mean value while θ_a is disordered for $K_a < 1/2$ and the situation is reversed for $K_a > 1/2$. As a consequence four different phases exists, similarly to the spin one case¹⁵, regardless of the sign of the interchain coupling. Three of these four phases correspond to the one derived in Ref. 16,44, but, as discussed in detail below an additional massless phase can exist. For isotropic intrachain coupling the results of Ref. 18 are of course recovered. A summary of the phase diagram is shown on Figure 1. Let us now discuss each phase in details. The physics of these phases depends whether interchain coupling is ferro or antiferromagnetic, since this changes the average value taken by the massive fields.

A. Ferromagnetic interchain coupling

A summary of the physics of each phase can be found in Table I. A qualitative understanding of the results of table I can be obtained in the limit $|J_\perp| \gg J, J_z$. Then, the ground state energy is minimized by forming a spin one on each rung with the 2 spins $1/2$. That case is thus physically identical to the spin one antiferromagnet¹⁵. The same physics hold even if $J_\perp \ll J, J_z$. As shown in Table I four sectors exist.

In Sector I, the two spins $1/2$ on every rung point in the same direction parallel to the z axis while each chain is antiferromagnetically ordered giving an effective spin one Ising Antiferromagnet.

In sector II a completely gapped phase exists where all spin-spin correlation functions decay exponentially. Using the analogy with the antiferromagnetic spin 1 chain, that phase is identified with a Haldane gap phase¹⁴. Singlets are formed here along the *legs* of the ladder⁴⁵. The identification can be made more complete by proving that the VBS order parameter, characteristic of a Haldane phase, is effectively non-zero (see appendix B).

Sector IV is a phase where only XY correlations remains. This XY1 phase has order parameter $S_1^+ + S_2^+ = S^+$ in agreement with the physical picture of a total spin equal to one. Semiclassically, the XY1 phase tends to have the two spins in the XY plane. This phase has only a gap in the antisymmetric spin excitations and has gapless symmetric excitations. (See table I, sector IV for the bosonized expression of the order parameter).

Finally, in sector III, $S_1^+ S_2^+$ which can be viewed as an effective raising operator for a spin $1/2$ is an order parameter. Its bosonized expression can be found in table I. In the case of a true spin 1 chain¹⁵, such a phase can be obtained by adding an anisotropy of the form $-D(S_1^z + S_2^z)^2$ which favors the states $S^z = +1, -1$ over $S^z = 0$. Thus, the XY2 phase is a phase in which the two spins on each rung can only be found with either $(S_1^z, S_2^z) = (1/2, 1/2)$ or $(S_1^z, S_2^z) = (-1/2, -1/2)$. They can make transitions from one of these state to the other and therefore form an effective spin $1/2$. This accounts for the critical fluctuations in that phase. The XY2 phase also shows subdominant Ising Antiferromagnet fluctuations due to the existence of the intrachain antiferromagnetic coupling. All other single spin fluctuations decay exponentially fast.

At small J_\perp , (11) gives $K_s \sim K_a$. Thus, only the Haldane gap, XY1 and Ising Antiferromagnet phase can be reached. These three phases were studied in Ref. 16,44. For isotropic coupling refermionization indeed gives an Haldane phase¹⁸. When J_\perp is very large the effective Hamiltonian for the resulting spin one chain contains only J_{xy}, J_z terms, and thus only the same phases can be reached. However, since when J_\perp increases, (11) indicate that K_a decreases, whereas K_s increases it is interesting to investigate whether one could reach the XY2 phase an intermediate J_\perp . Of course answering this question probably needs a numerical study of the model, since all the couplings would be of the same order. Even if this interesting possibility does not occur, the XY2 phase can always be obtained by using more complicated interchain couplings.

Finally, it is noteworthy that the only phase with SU(2) symmetry is the Haldane gap phase. Two ferromagnetically coupled isotropic spin 1/2 chains, thus always form a Haldane gap phase, irrespective of the strength of the coupling. The other phases are characteristic of anisotropic systems.

B. Antiferromagnetic interchain coupling

Results for the antiferromagnetic coupling are summarized in table II.

Sector I is again an Ising antiferromagnet phase. The spins on the same rung point in opposite directions so that in lattice representation, the order parameter is $(-)^n(S_{n,1}^z - S_{n,2}^z)$. The boson representation of this order parameter can be found in table II.

Sector IV corresponds to the XY1 phase obtained in Ref. 16. Its order parameter is $S_1^+ - S_2^+$. See table II for the bosonized expression. In that phase, the spins of each rung stay in the XY plane and are antiparallel. They are free to rotate around the z-axis provided they remain antiparallel and are antiferromagnetically coupled to their neighbors giving rise to an XY phase.

Sector III is the other XY phase, the XY2 phase, for which the order parameter is $S_1^+ S_2^+$. The hidden spin 1/2 degree of freedom corresponds now to the two states of zero z component of the spin formed by the addition of the spins 1/2 on the same rung. To the best of our knowledge, such a phase was not investigated in the previous studies^{44,18}. The XY2 phase presents subdominant Ising Antiferromagnet fluctuations. Thus, in the XY2 plane, there is a tendency to have the 2 spins parallel to the z axis and antiparallel among themselves. The intrachain coupling allows transitions between these two states leading to critical fluctuations analogous to the one of a spin 1/2 system.

Finally, there is a Singlet phase in which there is a gap to all excitations and all correlation functions of spin operators decay exponentially fast. A simple picture of that phase is obtained in the limit $J_\perp/J_z \rightarrow \infty$. In that limit, the ground state can be seen as made of singlets along the rungs. In such a phase, there is an effective zero spin on each site and accordingly all the spin-spin correlation function are zero. For finite J_\perp/J_z , massive triplet excitations can propagate along the chain leading to exponentially decaying spin-spin correlation functions.

As in the case of a ferromagnetic interchain coupling, at small J_\perp/J_z the only possible phases are the XY1 phase, the singlet phase and the Ising Antiferromagnet phase analyzed in Ref. 16. The XY2 phase appears at $K_s > 1$ and $K_a < 1/2$ (see table II) and thus,

as for the ferromagnetic case, cannot be obtained for purely local intrachain coupling and small J_{\perp} . Moreover, for antiferromagnetic interchain coupling, increasing J_{\perp} does not favor the formation of the XY2 phase, so it is likely that one needs here longer range interchain coupling. Finally, it is important to remark that as in the case of the ferromagnetic interchain coupling, the only phase that has spin rotational symmetry is the singlet phase, and that all other phases can only exist in systems with anisotropic interactions.

IV. RANDOM FIELDS AND RANDOM PLANAR EXCHANGE

Since we now have a complete description of the phases in the pure system, the effects of various type of disorder on these phases can be considered. Very different physical effects occur depending on whether the disorder respects or breaks the $U(1)$ symmetry (i. e. the rotational symmetry around the z -axis) of the system). For the case of symmetric couplings in the XY plane for the pure system the first type of disorder is the one most likely to occur physically. We consider its effects in the present section. As we discuss in more detail later perturbations that break the $U(1)$ symmetry are nevertheless also interesting to study and are considered in section V.

A. Coupling to disorder

Two type of $U(1)$ conserving disorder can be investigated. The first one is the standard random field along the z direction, while the other is the random exchange. If a representation of the spins in terms of hard core bosons was used such perturbations would correspond to random potentials or random hopping along the chains for the associated disordered boson problem⁴⁶.

1. Random field along z

Let us consider first the coupling to a random magnetic field along the z direction. This field is assumed weak enough not to destroy the gaps of the pure system. The lattice Hamiltonian for that problem is $H = H_{\text{pure}} + H_{\text{ZF}}$ with

$$H_{\text{ZF}} = \sum_i \left[h_{i,1}^z S_{i,1}^z + h_{i,2}^z S_{i,2}^z \right] \quad (12)$$

For simplicity the disorder is taken Gaussian and uncorrelated from chain to chain and from site to site:

$$\overline{h_{i,1}^z h_{i,2}^z} = 0 \quad (13)$$

$$\overline{h_{i,n}^z h_{j,n}^z} = D \delta_{i,j} \quad (14)$$

First, a bosonized representation of (12) has to be obtained. To do so, one goes to the continuum limit, decompose the h^z fields in a slowly varying ($q \sim 0$) and an oscillating part ($q \sim 2k_F = \frac{\pi}{a}$), introduce the bosonized expression for spin operators (4), and keep only in

the coupling to disorder the slowly varying terms, in a way similar to Ref. 8,9. The final result in terms of the ϕ_1 and ϕ_2 fields is :

$$H_{ZF} = \int dx \left[h_1^{2k_F}(x) \cos 2\phi_1(x) + h_2^{2k_F}(x) \cos 2\phi_2(x) \right] - \int \frac{dx}{\pi} \left[h_1^{q \sim 0}(x) \partial_x \phi_1(x) + h_2^{q \sim 0}(x) \partial_x \phi_2(x) \right] \quad (15)$$

with

$$\overline{h_i^{2k_F}(x) h_j^{2k_F}(x')} = D_{2k_F} a \delta(x - x') \delta_{i,j} \\ \overline{h_i^{q \sim 0}(x) h_j^{q \sim 0}(x')} = D_{q \sim 0} a \delta(x - x') \delta_{i,j} \quad (16)$$

All other correlators are zero. Introducing the symmetric and antisymmetric combinations $h_s^{q \sim 0}(x) = h_1^{q \sim 0}(x) + h_2^{q \sim 0}(x)$, and $h_a^{q \sim 0}(x) = h_1^{q \sim 0}(x) - h_2^{q \sim 0}(x)$, (15) is rewritten in terms of the fields ϕ_a, ϕ_s as :

$$H_{ZF} = H_{ZF}^{q \sim 0} + H_{ZF}^{q \sim 2k_F} \quad (17)$$

$$H_{ZF}^{q \sim 0} = - \int \frac{dx}{\pi \sqrt{2}} \left[h_s^{q \sim 0}(x) \partial_x \phi_s(x) + h_a^{q \sim 0}(x) \partial_x \phi_a(x) \right] \quad (18)$$

$$H_{ZF}^{q \sim 2k_F} = \int dx \left[h_1^{2k_F}(x) \cos \sqrt{2}(\phi_a + \phi_s)(x) + h_2^{2k_F}(x) \cos \sqrt{2}(\phi_a - \phi_s)(x) \right] \quad (19)$$

(17) is the starting Hamiltonian for the discussion of the effect of a random z field in the framework of bosonization. (18) represents the coupling of the system with the $q \sim 0$ components of the random field, and (19) the coupling with the $2k_F$ components.

The $\partial_x \phi_s$ term in (18) has to be discussed first. Two cases have to be distinguished : when there is no gap in ϕ_s , a transformation⁸ $\phi_s \rightarrow \phi_s + \int^x dx' h_s^{q \sim 0}(x')$, results in a new Hamiltonian of the form (17) with $h_s^{q \sim 0} = 0$, and identical correlators for the disorder. Therefore, the $h_s^{q \sim 0}$ terms cannot affect the low energy spectrum of the system and only affect its correlation functions in a trivial way. On the other hand, if there is a gap in the ϕ_s degrees of freedom, through refermionization the problem in the ϕ_s sector is equivalent to a filled band of fermions with a gap to excitations in the upper band and a small slowly varying random chemical potential. The random potential being smaller than the gap, it cannot induce transitions in the upper band (i.e. the system is incompressible) and thus cannot disorder the system. Therefore, in both cases, at weak disorder strength the $h_s^{q \sim 0}$ is always irrelevant. A similar situation occurs for the $\partial_x \phi_a$ term in (18). When ϕ_a has a gap the problem is obviously identical to the problem of the gap in ϕ_s . When θ_a is massive, the refermionization procedure shows that computing the response to the $\partial_x \phi_a$ term amount to computing the superconducting response function of a band insulator. Clearly, this has an exponential decay so that the random field coupled to $\partial_x \phi_a$ term cannot disorder the system. Thus all the $q \sim 0$ terms (18) have no important effect on the low energy response of the 2 chain system and can be discarded. If the disorder is stronger however, the $q \sim 0$ part of the disorder will however be able to destroy the gaps of the pure system. This case will be discussed later.

Having shown that the $q \sim 0$ do not affect the phase diagram at weak disorder, the $2k_F$ terms (19) remain to be treated . To simplify the notations the shorthand h_1 is used

for $h_1^{2k_F}$. The effect of the $2k_F$ disorder depends on the phase that would exist in the pure system in the absence of disorder. A detailed phase by phase discussion of the effects of $2k_F$ randomness is given in section IV B. The phase diagram in presence of disorder is shown in figure 2.

2. Random exchange

The other type of disorder that preserves the rotational invariance around the z axis is random planar exchange and random z exchange. For convenience the random z exchange and random planar exchange are treated separately.

Restricting to the small disorder case, the Hamiltonian in the planar exchange case is $H = H_{\text{pure}} + H_{\text{PE}}$ with

$$H_{\text{PE}} = \sum_n \left[J_n^1 (S_{n,1}^x S_{n+1,1}^x + S_{n,1}^y S_{n+1,1}^y) + J_n^2 (S_{n,2}^x S_{n+1,2}^x + S_{n,2}^y S_{n+1,2}^y) \right] \quad (20)$$

with $\overline{J_n^p J_{n'}^{p'}} = D \delta_{n,n'} \delta_{p,p'}$ $p = 1, 2$ being the chain index and $J_n^p \ll J$. The bosonized Hamiltonian is given by⁹

$$\begin{aligned} H_{\text{PE}} = & \int dx \left[J_1(x)^{2k_F} \sin 2\phi_1(x) + J_2(x)^{2k_F} \sin 2\phi_2(x) \right] \\ & + \int dx \frac{v_F}{2\pi} J_1(x)^{q \sim 0} \left[(\pi \Pi_1)^2 + (\partial_x \phi_1)^2 \right] \\ & + \int dx \frac{v_F}{2\pi} J_2(x)^{q \sim 0} \left[(\pi \Pi_2)^2 + (\partial_x \phi_2)^2 \right] \end{aligned} \quad (21)$$

where $J_p(x)$ is the continuum limit of J_n^p .

Let us now consider the random z exchange case. In the lattice spin 1/2 representation, the coupling to disorder is represented by

$$H_{\text{random } J_z} = \sum_i \left[J_{i,1} S_{i,1}^z S_{i+1,1}^z + J_{i,2} S_{i,2}^z S_{i+1,2}^z \right] \quad (22)$$

The bosonized form for the $q \sim 0$ term is easily obtained in the form

$$H_{\text{random } J_z}^{(q \sim 0)} = \int dx \left[J_z^{q \sim 0,1}(x) (\partial_x \phi_1)^2(x) + J_z^{q \sim 0,2}(x) (\partial_x \phi_2)^2(x) \right] \quad (23)$$

Obtaining the $2k_F$ is not so straightforward. The derivation is given in Appendix C. The final result is

$$H_{\text{random } J_z} = \int dx \left[J_z^{2k_F,1}(x) \frac{\sin 2\phi_1(x)}{2\pi a} + J_z^{2k_F,2}(x) \frac{\sin 2\phi_2(x)}{2\pi a} \right] \quad (24)$$

with $\overline{J_z^{2k_F}(x) J_z^{2k_F}(x')} = D \delta(x - x')$. The breakdown of bosonization for too large random planar exchange can be read off in the $q \sim 0$ terms: if $q \sim 0$ disorder becomes too large the quadratic part of the Hamiltonian is no more positive definite leading to a breakdown of the bosonization description. However, if the weak disorder condition is met the $q \sim 0$ terms couple to Π^2 and $(\partial_x \phi)^2$ terms. Power counting then implies that the $q \sim 0$ terms are

irrelevant. Thus, the stability against weak random exchange is again determined only by the $2k_F$ terms.

Bosonization leads to the same Hamiltonians (21) and (24) for the coupling with the $2k_F$ part of random planar exchange and random z exchange. These two perturbations can thus be treated the same way. In addition, in the phases in which ϕ_s is gapless, one can make the transformation:

$$\begin{aligned}\phi_1 &\rightarrow \frac{\pi}{4} + \phi_1 \\ \phi_2 &\rightarrow \phi_2 + \pi/4\end{aligned}\tag{25}$$

changing the random exchange term into a random z field as can be seen by comparing (24) and (21) for the random exchanges and (19) for the random field. Thus in the XY1 and XY2 phases both perturbations lead to the same stability regions and the same correlation lengths as is discussed in Section IV B. Of course the physical properties of the disordered phase are different depending whether the disorder is random exchange or random field. The phase diagram in the presence of random exchange is given in Figure 4.

B. Effect of disorder

In this section the effects of both the random z -field and the random exchange are considered in detail. The simpler case of the XY phases is discussed first, then the case of the Haldane gap phase and finally the case of the Antiferromagnet phase.

1. XY2 phase

The XY2 phase i. e. The X-Y phase with non-zero mean value of ϕ_a (sector III of tables I and II) is considered. For the sake of definiteness, only the random z field case is analyzed, but as discussed before in the XY phases the same results also apply in the random exchange case. Using (19) the effective Hamiltonian describing the coupling with the impurity potential is

$$H_{ZF} = C \int dx (h_1(x) + h_2(x)) \cos(\sqrt{2}\phi_s)(x) \quad \text{for } J_\perp < 0 \tag{26}$$

$$H_{ZF} = C' \int dx (h_1(x) + h_2(x)) \sin(\sqrt{2}\phi_s)(x) \quad \text{for } J_\perp > 0 \tag{27}$$

where $C = \langle \cos \phi_a \rangle$ and $C' = \langle \sin \phi_a \rangle$ (see table I and table II). The relevance of disorder can be determined by looking at the renormalization of the disorder term⁸.

The RG equation for disorder is obtained from the scaling dimensions of operators entering the coupling with disorder as it is the same for both (26) and (27).

$$\frac{dD^{2k_F}}{dl} = (3 - K_s)D^{2k_F}(l) \tag{28}$$

Note that the RG equation for the disorder does not depend on the nonuniversal constants C , C' . The RG equation for the Luttinger liquid parameter K_s is :

$$\frac{dK_s}{dl} = -\Lambda^2 D \quad (29)$$

where $\Lambda = C, C'$ depending on $\langle \phi_a \rangle$. This equation is non-universal. However, for very weak disorder the region of stability is given by (28), and one can discard (29) except to compute the critical properties very close to the transition⁸. Even for finite disorder (28) gives correctly the renormalized value of the parameter K , and thus the exponents of the correlation functions at the transition.

Equation (28) immediately shows that the XY2 phase is unstable unless $K_s > 3$. It also gives the correlation length in the disordered phase, for weak disorder and if one is not too close to the transition point⁸. The renormalization of K_s at weak disorder, (29) being negligible, the RG equation for D can be integrated into $D(l) = D(0)e^{(3-K_s)l}$. This form of $D(l)$ is valid as long as $D \ll \frac{v_F^2}{\alpha}$, since this condition ensures that the energy scale induced by disorder is much smaller than the energy cutoff. For $D(l) \sim \frac{v_F^2}{\alpha}(0)$ disorder cannot be treated as a perturbation and the RG has to be stopped. At that lengthscale, the system appears to be strongly disordered. Under the RG flow, the cut-off length has increased from $\alpha(0)$ to $\alpha(l) = e^l \alpha(0)$ at which point the system appears to be strongly disordered. Below it the system is described by bosonization and disorder can be treated as a perturbation. Thus the length $\alpha(l)$ is the correlation length ξ , leading to

$$\xi = \left(\frac{1}{D} \right)^{\frac{1}{3-K_s}} \quad (30)$$

Using the transformation (25), the same region of instability $K_s < 3$ and the same correlation length $\xi = (1/D)^{\frac{1}{3-K_s}}$ are obtained for the random exchange case.

2. XY1 phase

The case of the XY1 phase in the presence of a random z field is now considered. This case is more involved since to first order the effective coupling with disorder is zero (see tables II, I, sector IV). However, the coupling to disorder although irrelevant by itself can generate relevant terms in higher order in perturbation. Such terms can be computed, as was done in the case of the two chain of fermions problem^{35,36}, by integrating over the massive modes. This leads to the effective Hamiltonian for the coupling of the massless modes ϕ_s to disorder

$$H_{\text{ZF,eff.}} = \int dx \eta_{\text{eff}}(x) \cos(\sqrt{8}\phi_s(x)) \quad (31)$$

with $\overline{\eta_{\text{eff}}(x)\eta_{\text{eff}}(x')} \propto D^2 \delta(x - x')$. By using the same method than above, the scaling dimension of the disorder is now $3 - 4K_s$, implying that the disorder is irrelevant unless $K_s < \frac{3}{4}$. But at $K_s = 1$ the g_2 term becomes relevant and drives the system towards the singlet (if $g_2 > 0$) or the Haldane gap (if $g_2 < 0$) phase (see tables II,I). Thus, the XY1 phase is unaffected by weak random z -fields (or using the transformation (25) by random exchange) since these perturbations are irrelevant in its whole domain of existence. The XY1 phase is thus unaffected by **all** the perturbations respecting the rotation symmetry around the z axis.

3. The gapped phases

From section IV B 2 one sees easily that a two leg ladder system in a singlet or a Haldane gap phase coupled to random z fields is described by the following effective Hamiltonian coming from integration of the massive antisymmetric modes

$$H = \int \frac{dx}{2\pi} \left[u_s K_s (\pi \Pi_s)^2 + \frac{u_s}{K_s} (\partial_x \phi_s)^2 \right] + \int dx \frac{2g_2 + \eta_{\text{eff.}}(x)}{(2\pi\alpha)^2} \cos \sqrt{8}\phi_s \quad (32)$$

In the above discussion of the XY1 phase, it was shown that the non-random g_2 term is more relevant than the disorder term. Thus if the disorder is weak enough the deterministic part of the $\cos \sqrt{8}\phi_s$ term dominates over the random one and the gapped phases are stable with respect to small random z -fields or small random exchange. Such stability is reasonable on physical grounds in a singlet phase. The stability to stronger disorder will be discussed in Section IV C 3.

4. The Ising Antiferromagnet

The Ising Antiferromagnet phase (sector I in tables I,II) is the simplest to discuss due to its classical character. However, a marked difference appears depending on whether one considers a random z field or a random exchange. For $J_\perp < 0$ one has $\langle (S_1^z + S_1^z) \rangle \neq 0$ and for $J_\perp > 0$, $\langle (S_1^z - S_1^z) \rangle \neq 0$. For the random z -field, a simple Imry-Ma type argument⁴⁰ shows that the long range order is lost and that the ground state of the disordered 2 chain system is made of domains of characteristic size $\sim 1/D^{2k_F}$. This result is identical to the case of a classical antiferromagnet. It can also be obtained directly on the spin Hamiltonian without using bosonization. The Ising Antiferromagnet phase of the two leg ladder system is thus unstable in the presence of an arbitrarily weak random z field, as was the case in the one chain problem^{9,40}. On the other hand, in the case of a weak random exchange, there is no coupling at all to the disorder and the Ising Antiferromagnet is *stable* in the presence of disorder.

A summary of the phase diagrams for the random z field and random exchange are respectively shown in Figure 2 and Figure 4.

C. Discussion and Physical properties

1. Comparison with a single chain

It is interesting to compare the above results with a single disordered spin chain. For a single chain only two phases exist in the pure system: the Ising antiferromagnet (for $K < 1/2$) and an XY phase (for $K > 1/2$) that is the analogous of the XY1 phase for the ladder system. The XY phase is destroyed by a random field along z or random exchange for $K < 3/2$ ^{8,9,47}. The antiferromagnetic phase is unstable in the presence of a random magnetic field but not random exchange as shown by an Imry-Ma argument^{9,40}. The isotropic Heisenberg point is thus unstable to infinitesimal disorder.

For the ladder case the isotropic point corresponds now to a singlet gapped phase. Quite naturally this phase is insensitive to small disorder. The antiferromagnetic phase gives identical results than for a single chain. Quite remarkable however the XY1 **massless** phase is now resistant to all perturbations respecting around the z axis, at the opposite of the corresponding phase for the single chain. This surprising result can be understood by noticing that the XY1 phase is much more anisotropic than its one chain counterpart (in particular although it is massless it still has **exponentially** decaying $2k_F$ correlations of the S_z component). The disorder can only couple to higher operators that are less relevant, in a way reminiscent of the situation in the fermion ladder problem^{35,36}.

The opposite situation occurs for the XY2 phase. This phase reveals itself much more unstable than the XY phase in the presence of random z fields or random exchange. Its domain of existence is reduced to $K_s > 3$. As a result, its correlation length $l_{2ch.}^{XY2} \sim \left(\frac{1}{D}\right)^{\frac{1}{3-K_s}}$ is also much shorter than its single chain counterpart $l_{1ch.} \sim \left(\frac{1}{D}\right)^{\frac{1}{3-2K}}$. This is due to the existence of strong antiferromagnetic fluctuations in the XY2 phase that easily couple to disorder. It is again reminiscent of the easy coupling of the charge density wave phase for the fermionic ladder^{35,36}. The two most remarkable effects occurring in the ladder system are thus the stabilization of the isotropic point due to the singlet phase and quite unexpectedly the stability of the XY1 massless phase as well. This remarkable stability against disorder prompts for several questions. In particular it is interesting to understand how the phase diagram evolves if the gaps allowing for this stability against disorder are destroyed, either by a magnetic field or by increasingly strong disorder. These two cases are examined in the next two sections.

2. Effect of a magnetic field

The coupling to a uniform magnetic field has the form:

$$H_{\text{uniform}} = \frac{h\sqrt{2}}{\pi} \int dx \partial_x \phi_s \quad (33)$$

Therefore, a strong enough magnetic field, suppresses the gap formation in ϕ_s but does not affect the antisymmetric sector. Thus the antiferromagnetic phase and the Haldane gap phase disappear upon application of a strong enough magnetic field and are replaced by incommensurate phases⁴⁸. The stability of the remaining two incommensurate gapless XY phases in the presence of the random magnetic field or the random exchange can be analyzed by similar techniques than the ones of section IV B.

Using the analysis of Sec. IV B 2, one finds that the random z field results in the suppression of the XY1 phase for $K_s < 3/4$ and formation of a random antiferromagnet with a correlation length $l_{AF,4k_F} \sim \left(\frac{1}{D}\right)^{\frac{2}{3-4K_s}}$. The modulation of the spin density is at a wavevector $4k_F = 2\pi(1 + 2m)$, where m is the magnetization and is incommensurate⁴⁸ with the lattice spacing. Contrarily to the zero magnetic field case, the disordered $4k_F$ antiferromagnet is no more wiped out by Haldane gap formation. According to Sec. IV B 1, a random z field also results in suppression of the XY2 phase for $K_s < 3$ and formation of a random $2k_F$ antiferromagnet. The situation in this case is however identical to the zero magnetic

field case. The results are summarized on figure 5. In particular, for $1/2 < K_s < 3/4$ the system is again sensitive to disorder. The case of a random exchange is similar except that the $4k_F$ antiferromagnet is replaced by a $4k_F$ random singlets phase.

Using the standard mapping of spins on hard core bosons, the problem of the coupled chains under field can be related to the more general problem of a bosonic ladder. This problem will be analyzed in details elsewhere⁴⁶. In the boson language, the XY phases correspond to superfluid phases, and antiferromagnetic phases to charge density waves. Random z fields correspond to random potentials and random planar exchange to random hopping.

3. Stability to strong disorder

It is also interesting to look on how the gaps in the system can be destroyed by the disorder itself, when it becomes strong enough. Strictly speaking such a study is beyond the reach of a renormalization group treatment since such a transition would correspond to a transition between two strong coupling fixed points, in a regime where the RG is not rigorously applicable. Fortunately one can still make some physical arguments. It would be thus of great practical interest to check whether the simple analysis performed here can be confirmed in more sophisticated treatments such as simulations or by using non-perturbative methods^{49–51,34,52}.

For simplicity we concentrate on the Haldane (or singlet) phases and on the XY1 phase in the presence of a random magnetic field. Both the symmetric and antisymmetric gap, when they exist can be destroyed by either the $q \sim 0$ or $q \sim 2k_F$ component of disorder. A similar effect has been analyzed in the case of non-magnetic impurities in Ref. 53 using bosonization and RG techniques. The $q \sim 0$ component of the magnetic field can prevent ϕ_a from developing a gap, however if θ_a develops a gap, (i. e. $K_a > 1/2$) this component of the random magnetic field has simply no effect for the reasons exposed in Sec. IV A.

Three regimes of stability can thus a priori be defined. At very small J_\perp one has disordered decoupled spin $1/2$ chains where no gap exists both in the symmetric and antisymmetric sector. The correlation length in this regime is $\xi_{\text{decoupled}} \sim (1/D)^{1/(3-2K)}$, using the approximation $K_s \sim K_a \sim K$ for small J_\perp . This phase is unstable when the correlation length due to the opening of one of the gaps by J_\perp becomes comparable to $\xi_{\text{decoupled}}$. It is easy to check that the shortest correlation length induced by J_\perp is the antisymmetric one. The transition occurs therefore when

$$J_\perp^* \sim D^{(2-1/2K)/(3-2K)} \quad (34)$$

thus giving $J_\perp^* \sim D^{1/2}$ for $K = 1/2$ (the isotropic point) and $J_\perp^* \sim D^{3/2}$ for $K = 1$ (the limit to the XY1 phase). For larger values of J_\perp a gap exists in the antisymmetric θ_a mode.

Since the gap in the symmetric mode behaves as $J_\perp^{1/(2-2K)}$, the properties of this mode crucially depends on the value of K . The critical value of J_\perp needed to resist the $2k_F$ and the $q \sim 0$ of the random z field of the disorder are respectively

$$J_{\perp,c1} \sim D_{q \sim 0}^{2-2K_s} \quad (35)$$

$$J_{\perp,c2} \sim D_{q \sim 2k_F}^{\frac{4-4K_s}{3-4K_s}} \quad (36)$$

For the isotropic point $K = 1/2$ both these values are smaller than J_\perp^* . By increasing J_\perp one should thus go directly from two disordered decoupled spin 1/2 chains to a stable Haldane (or singlet) ladder system. Although one can only give physical arguments for the nature of the transition, the crossing of two correlation length suggest that there is a sudden drop of the gap to zero. It is clear that the decoupled chains do not have any form of topological hidden order and that their response function is simply the one of isolated disordered chains. The phase diagram is shown on Figure 3-a.

As one make the system more anisotropic, and moves towards the XY1 phase the gap in the symmetric sector decreases. The critical values of $J_{\perp,c1}$ become larger than J_\perp^* for $K \geq 0.76465$. For such values of K , the $2k_F$ component of the random z-field is irrelevant. Therefore, for $J_\perp^* < J_\perp < J_{\perp,c1}$ a gap is generated by interchain coupling in the antisymmetric mode but not in the symmetric mode due to the $q \sim 0$ component of the random z field. In other words, an intermediate gapless XY1 phase exists between the decoupled chain phase and the stable Haldane phase. Since the VBS order parameter is the correlation function $\langle \cos \sqrt{2}\phi_s(x) \cos \sqrt{2}\phi_s(x') \rangle$ in the limit $|x - x'|$ (see appendix B) it is strictly zero in the phase that results from the breaking of the Haldane gap, due to the presence of the forward scattering disorder. This situation is shown in Figure 3-b. If $K > 1$, J_\perp is irrelevant in the symmetric sector and one recovers a direct transition between the decoupled chains and the stable XY1 phase, with algebraic correlations, as shown on Figure 3-c.

For random exchange, it has been shown in Sec. IV A 2 that the $q \sim 0$ component of disorder is irrelevant in the RG sense. For values of K in the interval $[0.76465, 1]$ the $q \sim 2k_f$ part of the disorder is irrelevant, so random exchange which does not give rise to a relevant $q \sim 0$ term will continue to give the transitions of Figure 3-a. For $K > 1$, random exchange disorder gives the transitions of Figure 3-c.

It is noteworthy that the resulting phase diagrams for disorder larger than the gaps in the ladder system is quite different from the one where **no** coupling of the antisymmetric mode to the disorder could occur such as for a true spin one chain. In that case increasing the disorder would lead to a disordered Haldane phase that could retain some degree of VBS order for the random exchange case^{31,32}.

V. RANDOM PERTURBATION BREAKING THE U(1) SYMMETRY

From a theoretical point of view it is also interesting to consider randomness that break the $U(1)$ rotational symmetry of the XXZ model. Indeed for the single chain case⁹, such disorder was proved to be very efficient in destroying the quasi long range order in the system. Given the remarkable stability of the gapless phase of the ladder system to the type of disorder examined in section IV, it is interesting to check whether the same property still occurs. In this section, the two most common types of disorder breaking the $U(1)$ symmetry are considered, namely a random field confined to the XY plane and random planar anisotropy.

A. Coupling to disorder

1. Random field in the XY plane

The two chain system with a random magnetic field in the XY plane is considered. The Hamiltonian is $H = H_{\text{pure}} + H_{\text{XYF}}$, H_{XYF} being given by

$$H_{\text{XYF}} = \sum_i \left[h_{i,1}^x S_{i,1}^x + h_{i,1}^y S_{i,1}^y + h_{i,2}^x S_{i,2}^x + h_{i,2}^y S_{i,2}^y \right] \quad (37)$$

With $\overline{h_{i,p}^a h_{j,q}^b} = D \delta_{i,j} \delta_{p,q} \delta_{a,b}$ ($a, b = x, y$, $p, q = 1, 2$).

To bosonize this expression, it is convenient to rewrite (37) in terms of $S_{i,a}^\pm$, ($a = 1, 2$) and introduce $h_{i,a}^\pm = h_{i,a}^x \pm i h_{i,a}^y$. This gives

$$H_{\text{XYF}} = \frac{1}{2} \sum_i \left[h_{i,1}^+ S_{i,1}^- + h_{i,1}^- S_{i,1}^+ + h_{i,2}^+ S_{i,2}^- + h_{i,2}^- S_{i,2}^+ \right] \quad (38)$$

For chain 1, upon bosonization the coupling with the random XY field can be rewritten as⁹:

$$H_{\text{XYF},1} = \int \frac{dx}{\sqrt{8\pi a}} \left[h_1^{(q \sim 0)+} e^{-i\theta_1(x)} \cos 2\phi_1(x) + h_1^{(2k_F)+} e^{-i\theta_1(x)} \right] + \text{H. c.} \quad (39)$$

A similar expression holds for chain 2. From power counting, the most relevant terms in the bosonized Hamiltonian are the $2k_F$ ones. One could be tempted to keep only the $2k_F$ terms and simply drop the $q \sim 0$ ones. However, the $q \sim 0$ part of the coupling to disorder has a bosonized form $e^{\pm i\theta} \cos(2\phi)$ as can be seen from (4) and generates in second order in perturbation theory a term $\int dx h_{\text{eff},1}^z(x) \cos(2\phi_1(x))$ i.e an effective random magnetic field parallel to the z axis (see (4)) with $h_{\text{eff},1}^z(x) \propto h_1^{(q \sim 0)+} h_1^{(2k_F)-} + \text{h. c.}$. That term is always more relevant than the $q \sim 0$ term, and can be relevant even when the $2k_F$ term is irrelevant. It is easily seen that no other relevant terms are generated so that one has to keep just the generated random z field and drop the $q \sim 0$ term from the Hamiltonian. This situation is typical of random perturbations that break the rotation symmetry around the z axis⁹. The generated random z -field transforms the random field restricted to the XY plane into a more general anisotropically distributed random field. As a byproduct, the effect of an isotropic random field can also be easily obtained. It is obvious that the phase boundaries induced by the random XY field and the isotropic random field will be identical. However, the correlation length will not show the same dependence on disorder strength in the phases the properties of which are dominated by the z component of the random field.

From the preceding discussion of the generated terms, the bosonized Hamiltonian containing only the most relevant terms is $H = H_{\text{pure}} + H_{\text{XYF, bosonized}}$ with H_{pure} given by (10) and

$$H_{\text{XYF, bosonized}} = \int dx \left[\frac{h_1^{(2k_F)-}(x)}{\sqrt{2\pi a}} e^{i\frac{\theta_s + \theta_a}{\sqrt{2}}} + \frac{h_2^{(2k_F)-}(x)}{\sqrt{2\pi a}} e^{i\frac{\theta_s - \theta_a}{\sqrt{2}}} + \text{H. c.} \right] \\ + \int dx \left[h_{\text{eff},1}^z(x) \cos(\sqrt{2}(\phi_a + \phi_s)) + h_{\text{eff},2}^z(x) \cos(\sqrt{2}(\phi_a - \phi_s)) \right] \quad (40)$$

where $\overline{h_{\text{eff},p}(x) h_{\text{eff},p'}(x')} = CD^2 \delta(x - x') \delta_{p,p'}$. The second line of the above equation is the random z -field already studied in Section IV, the first line is the random planar field itself.

One can expect the phases showing strong XY fluctuations to be more strongly affected by the random XY field than the phases having dominant antiferromagnetic fluctuations. Indeed the latter ones only couple to the generated random z -field, or in other words their coupling to disorder involves intermediate states with energies above the gaps. Technically, this means that it is necessary to separate the case in which ϕ_a is massive (i. e. dominant antiferromagnetic fluctuations) and the case in which θ_a is massive (i. e. dominant XY fluctuations). Obviously, the singlet phase and the Haldane gap phase should be insensitive to small random fields since these phases are completely gapped and have vanishing spin-spin correlations.

2. Random planar anisotropy

Another form of perturbation breaking the rotational symmetry around the z axis is when $\delta J = J_X - J_Y$ fluctuates randomly from site to site with a mean value zero. The relevant Hamiltonian is $H_{\text{pure}} + H_{\text{RA}}$ in which

$$H_{\text{RA}} = \sum_i \left[\delta J_i^1 (S_{i,1}^x S_{i+1,1}^x - S_{i,1}^y S_{i+1,1}^y) + \delta J_i^2 (S_{i,2}^x S_{i+1,2}^x - S_{i,2}^y S_{i+1,2}^y) \right] \quad (41)$$

With $\overline{\delta J_i^p \delta J_j^q} = D \delta_{i,j} \delta_{p,q}$ ($p, q = 1, 2$). As in the case of the random XY fields, contractions of the $q \sim 0$ and $q \sim 2k_F$ terms generate relevant terms⁹. The bosonized form of the $q \sim 0$ part is :

$$H_{\text{RA}}^{q \sim 0} = \int dx \gamma_1(x) \cos(\sqrt{2}(\theta_a + \theta_s))(x) + \int dx \gamma_2(x) \cos(\sqrt{2}(\theta_s - \theta_a))(x) \quad (42)$$

with $\gamma_p(x = na) \propto \delta J_n^p$, and the bosonized form of the $2k_F$ part is :

$$\begin{aligned} H_{\text{RA}}^{2k_F} &= \int dx \gamma_1^{2k_F}(x) \cos(\sqrt{2}(\theta_a + \theta_s))(x) \cos(\sqrt{2}(\phi_a + \phi_s))(x) \\ &+ \int dx \gamma_2^{2k_F}(x) \cos(\sqrt{2}(\theta_s - \theta_a))(x) \cos(\sqrt{2}(\phi_a - \phi_s))(x) \end{aligned} \quad (43)$$

An effective random z field is generated by contraction of the $q \sim 0$ with the $2k_F$ part as in the case of the random field in the XY plane. A Its expression is

$$H_{\text{generated}} = \int dx \left[h_1^z(x) \cos(\sqrt{2}(\phi_a + \phi_s)) + h_2^z(x) \cos(\sqrt{2}(\phi_a - \phi_s)) \right] \quad (44)$$

where $\overline{h_p^z(x) h_{p'}^z(x')} = CD^2 \delta(x - x') \delta_{p,p'}$. This random z field (44) is always more relevant than the $2k_F$ part. Therefore only (42) and (44) have to be kept in the bosonized Hamiltonian.

B. Physical Properties

Having worked out the bosonized representations of the perturbations that break the $U(1)$ symmetry, it is possible to analyze their effects on the phases of the system.

1. XY1 phase

This phase is expected to be the most unstable in the presence of a random XY field or a random anisotropy. Since the XY1 phase is not affected by a random z -field (see Section IV), the corresponding term can be safely dropped. θ_a is replaced by its average value $\langle\theta_a\rangle$ in (40) giving the following simplified expression of the coupling to the random XY field:

$$H_{\text{XYF, eff.}} \sim \int dx \left[\xi_{\text{eff.}}(x) e^{i\frac{\theta_s(x)}{\sqrt{2}}} + \text{H. c.} \right] \quad (45)$$

Where $\xi_{\text{eff.}}(x) = (\xi_1(x) + \xi_2(x))$ for $J_\perp < 0$ and $\xi_{\text{eff.}}(x) = i(\xi_1(x) - \xi_2(x))$ for $J_\perp > 0$ (see tables I,II).

In the case of the random anisotropy, the effective coupling in the XY1 phase has the form:

$$H_{\text{RA}} = \int \gamma_{\text{eff.}}(x) \sin(\sqrt{2}\theta_s(x)) dx \quad (46)$$

for $J_\perp < 0$ and:

$$H_{\text{RA}} = \int \gamma_{\text{eff.}}(x) \cos(\sqrt{2}\theta_s(x)) dx \quad (47)$$

for $J_\perp > 0$. ϕ_s being massless, the RG equation for the random XY field is derived from (45) in the form

$$\frac{dD}{dl} = \left(3 - \frac{1}{4K_s} \right) D \quad (48)$$

Equation (48) implies that the random XY field is relevant for $K_s > 1/12$. Since the XY1 phase only exists at $K_s > 1$ (see tables I,II) this phase is completely suppressed by an arbitrarily weak random magnetic field in the XY plane. At small disorder the correlation length of the disordered XY1 phase is

$$l_{\text{r. field}} \sim (1/D)^{\frac{4K_s}{12K_s-1}} \quad (49)$$

Similarly for the case of planar anisotropy, the RG equation for the disorder is

$$\frac{dD}{dl} = \left(3 - \frac{1}{K_s} \right) D \quad (50)$$

The disorder is thus relevant for $K_s > 1/3$ and the XY1 phase is also unstable to weak random anisotropy. The correlation length is in that case

$$l_{\text{r. anisotropy}} \sim (1/D)^{\frac{K_s}{3K_s-1}} \quad (51)$$

For the isotropic point $l_{\text{r. anisotropy}} \sim 1/D$.

2. Singlet and Haldane gap phases

In these two phases, the coupling with disorder has the same simplified expression (46-47) and (45) than in the XY1 phase due to the identical structure of the gaps in the antisymmetric modes. Moreover, it is known from section IV that a random z -field has no effect.

However, the presence of a mass term in the symmetric part of the Hamiltonian leads to the suppression of all disorder correlation functions. The gapped phase is thus stable as for the case of perturbations preserving the rotational symmetry.

3. The XY2 phase

This phase has weaker XY fluctuations than the XY1 phase and should therefore be less affected by the random XY field or the random anisotropy. Nevertheless, it has subdominant antiferromagnetic fluctuations and can be disordered by coupling to the generated random z field.

Indeed it is known that random z fields suppress the XY2 phase if $K_s < 3$ (see Section IV). For $K_s > 3$ on the other hand, only the random XY field or the random anisotropy can possibly disorder the XY2 phase. In the XY2 phase, ϕ_a is massive, and this gives apparently zero coupling to disorder when simplifying (40). In fact, an effective coupling of the random XY field to ϕ_s can be derived through second order perturbation theory along the lines of Ref. 35,36. The calculations are straightforward and lead to the following effective coupling to the random XY field

$$H_{\text{XYF, eff.}} = \int dx \left[\xi_{\text{eff.}}(x) e^{i\sqrt{2}\theta_s} + \text{H. c.} \right] \quad (52)$$

where $\overline{\xi_{\text{eff.}}(x)\xi_{\text{eff.}}(x')} \propto D^2\delta(x-x')$, and prefactors coming from mean values of the ϕ_a fields have been omitted. A similar calculation for the random anisotropy case leads to the following effective coupling:

$$H_{\text{RA, eff.}} = \int dx \gamma_{\text{eff.}}(x) \cos(\sqrt{8}\theta_s(x)) \quad (53)$$

With $\overline{\gamma_{\text{eff.}}(x)\gamma_{\text{eff.}}(x')} \propto D^2\delta(x-x')$.

From power counting, the RG equation for the random field is

$$\frac{dD_{XY}}{dl} = \left(\frac{3}{2} - \frac{1}{2K_s} \right) D_{XY} \quad (54)$$

implying that the $2k_F$ part of the random XY field is relevant as soon as $K_s > 1/3$. The correlation length in the disordered phase is

$$l_{\text{random field}} \sim (1/D_{XY})^{\frac{2K_s}{3K_s-1}} \quad (55)$$

For random anisotropy, the RG equation is:

$$\frac{dD_{RA}}{dl} = \left(\frac{3}{2} - \frac{2}{K_s} \right) D_{RA} \quad (56)$$

Since the random planar anisotropy is relevant for $K_s > \frac{4}{3}$, the XY2 phase does not survive the presence of a small random planar anisotropy also for $K_s > 3$. the correlation length of the resulting disordered phase is then given by $l_{\text{random anisotropy}} \sim (1/D_{RA})^{\frac{2K_s}{3K_s-4}}$.

4. The Ising antiferromagnet

Since the antiferromagnetic phase is unstable with respect to a weak random z-field (see section IV), it will be completely suppressed here by the generated random z field both in the case of random anisotropy and random field in the XY plane. The corresponding disordered phase is a random antiferromagnet of correlation length $l \sim 1/D_z$. This correlation length is just the one of the disordered XY2 phase “frozen” at $K = 1$.

The full phase diagram is given on figure 6.

C. Physical discussion

1. comparison with the one chain system

In order to compare the two XXZ spin chain system with a single XXZ chain, the discussion has to be restricted to the antiferromagnetic, singlet and XY1 phases since the XY2 phase does not obtain in a two XXZ chain system. The single chain system has a XY phase for $J_z < J_{XY}$ and an antiferromagnetic phase for $J_z > J_{XY}$. The XY phase of the single chain is suppressed by a random field in the XY plane and the correlation length is⁹ $l_{\text{1ch.}} \sim (1/D_{XY})^{\frac{2K}{6K-1}}$. The XY1 phase in the two chain system is also suppressed, but : $l_{\text{RF,2ch.}} \ll l_{\text{RF,1ch.}}$ (see (49)). Similarly, it is also unstable in the presence of random anisotropy⁹, with a correlation length $l_{\text{RA,1 ch.}} \sim (1/D_{RA})^{\frac{K_s}{3K_s-2}}$ in the disordered phase. Again this length is much larger than its ladder counterpart $l_{\text{RA,2ch.}} \ll l_{\text{RA,1ch.}}$ (see (51)). Thus, interchain coupling makes an XXZ ladder systems much more sensitive to random perturbations that break rotational symmetry around the z axis than a single chain. In both case the effect of disorder is considerably enhanced since the *exponents* of the correlation length are changed. By contrast, for perturbations that preserve the rotation symmetry around the z axis the one chain system was *more* disordered than its two chain counterpart. This seem to indicate that the XY phase in the ladder system is in effect much more anisotropic than its single chain counterpart.

As for perturbations respecting the rotation symmetry, effects of disorder in the antiferromagnetic phase are quite similar for a single chain and for the ladder. Indeed for a single chain, contrarily to the results of Ref. 9, the generated random z field suppresses the antiferromagnet phase, both in the case of a random field in the XY plane and of random anisotropy. The same effect also holds in the two chain system. In that case, the correlation lengths show the same dependence with disorder strength up to prefactors. Contrarily to the one chain case, there is no phase boundary between the disordered XY phase and the

random antiferromagnet phase due to the existence of the Haldane gap (or the singlet) phase which is not suppressed by a weak disorder.

The XY2 phase is also suppressed. For a random field, however, the correlation length in the disordered phase is much longer than the one of the disordered XY phase of a single chain, $l_{\text{corr.}} \sim (1/D)^{\frac{2K}{6K-1}}$. In the presence of random anisotropy, the XY2 phase is also suppressed. However, the XY2 phase gives rise to two different disordered phases. The first one corresponds to suppression of the quasi long range order by the random anisotropy per se, the other one to suppression of QLRO by the effective random z field. A crossover occurs between these two phases at the point where they have identical correlation lengths. The correlation length of the phase pinned on the random z field being given by $l_{\text{corr.}} \sim (1/D_z)^{\frac{2}{3-K_s}}$ where D_z measures the strength of the effective random magnetic field. Since $D_z \sim D^2$ the crossover should occur for $K_s = 2$. for $K_s < 2$, the properties are dominated by the generated random z field and for $K_s > 2$ by the random anisotropy. Two different disordered phases already existed in the XXZ chain with random planar anisotropy⁹. Such an effect in the two chain system is specific of the XY2 phase and does not exist in the XY1 phase. The correlation length of the disordered phase induced by a random anisotropy in a XY2 phase is larger than the correlation length of its one chain counterpart $l_{\text{corr.,1 ch.}} \sim (1/D)^{\frac{K}{3K_s-2}}$. Paradoxically, although the XY2 phase does not exist in an XXZ two chain system with weak antiferromagnetic coupling, its properties **regarding disorder** are much more similar to the one of the XY phase of the single XXZ chain than the ones of the XY1 phase. Such difference is explained by the fact that in the XY1 phase the $2k_F$ fluctuations in S^z are suppressed by gap formation leading to an increased robustness with respect to perturbations that respect rotational symmetry around the z axis and by the locking of the spins that sit on the same rung of the ladder leading to an increased sensitivity to perturbations that break rotation symmetry around the z axis. Also, it can be shown that in the XY2 phase, the spins on each rung of the ladder can only take identical values (for $J_{\perp} < 0$) or opposite values (for $J_{\perp} > 0$) leading to an effective spin 1/2 chain.

2. Effect of a uniform magnetic field

As seen in section IVC 2, a uniform magnetic field can inhibit Haldane gap formation. In the presence of a random XY field or random anisotropy, this lead to the formation of a disordered XY phase, with a correlation length given by Eq. (49) for a random XY magnetic field and by Eq. (51) for a random planar anisotropy. It can be verified that the correlation length induced by the effective random z field is much longer than the correlation length induced by the random XY field for $K_s > \frac{\sqrt{97}-9}{8} \simeq 0.106$ and much longer than the correlation length induced by the random XY anisotropy for $K_s > \frac{\sqrt{145}-9}{8} \simeq 0.3802$. In other words, the induced random z field is a very weak perturbation that takes over only when the random field in the XY plane or the random anisotropy is nearly irrelevant. Therefore, in the presence of an applied uniform magnetic field, a disordered phase replaces the Haldane gap phase. The resulting phase diagram can be found on figure 7 for the random magnetic field in the XY plane and on figure 8 for the random planar anisotropy.

VI. CONCLUSION

In this paper we have investigated the effects of disorder on an anisotropic two leg spin ladder.

For the pure system, we have computed the phase diagram in details. Both for ferromagnetic and antiferromagnetic interchain coupling the system can exhibit four different phases: an antiferromagnetically ordered state, a gapped singlet phase (Haldane phase) and two massless XY phases. One of the XY phases (XY1) is a close analogous to the XY phase of a single chain. The other one (XY2), that was not discussed in previous work on the 2 coupled chain system, is the analogous to the one occurring for a spin one chain with on site anisotropy. That phase has a tendency to have the spins parallel to the z axis. For weak interchain coupling only the AF, Haldane and XY1 phases can be realized. An interesting question is whether for intermediate coupling one can stabilize or not the XY2 phase. Of course such a phase can always be realized by using more complicated interchain couplings.

The disordered ladder shows remarkable features compared to a single spin chain. For perturbations respecting the XY symmetry such as a random z -field and random exchange, both the singlet phase and the **massless** XY1 phases revealed to be totally insensitive to weak disorder. On the other hand the XY2 phase is extremely sensitive to disorder due to the presence of strong antiferromagnetic fluctuations in the z direction. Similarly to the one chain case, the effect of randomness on the Ising antiferromagnet depends on whether the perturbation is invariant under $S^z \rightarrow -S^z$ or not. In the former case it does not affect the Ising AF in the latter it suppresses it through an Imry Ma mechanism. While such a stability was to be expected from the gapped singlet phase, it is much more surprising in the massless XY phase and is reminiscent of the delocalization occurring in fermionic ladders. Note that here the effect is even stronger, since the delocalization transition only occurred for weakly attractive interactions close to the non-interacting point. This would correspond to a non-disordered phase for $J_z < 0$ and a disordered phase for $J_z > 0$ in the spin language, whereas the spin system is stable with respect to disorder for $J_z < J$. In the presence of a finite magnetic field this stability to disorder subsides, even if the singlet phase is destroyed. The isotropic point $K = 1/2$ becomes now sensitive to disorder, but for moderate XY anisotropy $K > 3/4$ disorder has again no effect. This problem offers interesting connections to disordered bosonic ladders or coupled vortex planes. This extreme stability to disorder for the spin ladder is to be strongly contrasted with the one for a single chain where most of the phase diagram is destroyed by infinitesimal disorder and only extremely anisotropic XY chains $K > 3/2$ can resist.

Knowledge of the phase diagram for weak disorder allows to make reasonable guess on the behavior of the system for stronger disorder or when the gaps are reduced (e.g. by diminishing the interchain couplings). Close to the isotropic point $K \sim 1/2$, upon increasing the interchain coupling, there should be a single transition between a regime of decoupled (and thus disordered) spin $1/2$ chains towards a stable singlet phase ladder for $J_{\perp}^* \sim D^{(2-1/2K)/(3-2K)}$. However by making the system more XY anisotropic the transition should occur in two step. Decoupled spin chains would have a transition towards coupled chains, where the antisymmetric mode is gapped, but the symmetric mode is not gapped, leading to an XY1 phase. Upon increasing the interchain coupling further the symmetric mode gaps giving back the singlet phase. Thus for anisotropic ladders with $0.76465 < K <$

1 an intermediate **non-disordered** XY1 phase should appear for intermediate interchain couplings. For large anisotropies the singlet phase does not exist any more and one recovers a direct transition between decoupled chains and the stable XY1 phase of the ladder. Since this transition are obtained by the crude comparison of the various correlation length in the system it would be interesting to confirm such a phase diagram by numerical simulations. Subtle effects might indeed occur for disorder such as the random exchange where the true correlation length of the system (compared to the one given by the RG around the Gaussian fixed point) is found to diverge⁵¹.

The extreme stability of the ladder to XY symmetric randomness made it worth to investigate perturbations breaking this symmetry as well. Remarkably the behavior is here inverted. Randomness breaking the rotational symmetry around the z axis, suppresses the two XY phases and the Ising AF phase as was the case for one chain. However the disordered XY1 phase has a *much shorter* correlation length than the disordered XY phase of the one chain system in the presence of the same perturbations whereas the XY2 phase has a *much longer* correlation length than the XY phase of the one chain system. Such an effect is again due to the fact that the ladder system with a given $J_z/J \neq 1$ is in effect much more anisotropic than its one chain counterpart with the same J_z/J parameter. The XY2 phase in the presence of planar anisotropy gives rise to two different disordered phases similarly to the XY phase of the one chain system but at odds with the XY1 phase that gives rise to only one phase. Such result is at first sight paradoxical since the XY1 appears as the natural continuation of the XY phase of the one chain XXZ system, whereas the XY2 phase is likely to exist only in more complicated models or at larger coupling.

In the presence of a strong enough uniform magnetic field, the Haldane gap phase is suppressed, allowing for the observation of a crossover from a phase dominated by the random field in the XY plane or the random planar anisotropy to a phase dominated by a generated random field parallel to the z axis. This crossover occurs in the vicinity of the transition to the disordered classical antiferromagnet and may therefore be difficult to probe in experiments or numerical simulations.

Clearly the disordered ladder system presents an extremely rich behavior and the renormalization study presented here can only be a first step towards its understanding. This is specially true for the crossover to strong disorder, which would be interesting to investigate through numerical simulations and non perturbative methods.

ACKNOWLEDGMENTS

We are grateful to R. Bhatt, R. Chithra, P. Le Doussal, A. J. Millis, and H.J. Schulz for useful discussions. E. O. thanks ISI for hospitality and support for participating the 1996 Euroconference of “The role of Dimensionality in the Correlated Electron Systems” held in Torino May 6-25, 1996. T. G. thanks the ITP where part of this work was completed for support under NSF grant PHY94-07194.

APPENDIX A: SINE-GORDON HAMILTONIANS

The general sine-Gordon Hamiltonian has the form:

$$H_{SG} = \int \frac{dx}{2\pi} \left[uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x\phi)^2 \right] + \Delta \int dx \cos(\beta\phi) \quad (\text{A1})$$

To get some understanding of (A1), let us first put $\Delta = 0$. The simplified Hamiltonian is

$$H = \int \frac{dx}{2\pi} \left[uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x\phi)^2 \right] \quad (\text{A2})$$

Computing the correlation functions at 0K for (A2) in Matsubara time gives :

$$\langle T_\tau e^{in\phi(x,\tau)} e^{-in\phi(0,0)} \rangle = \left(\frac{x^2 + (u\tau)^2}{\alpha^2} \right)^{-\frac{n^2 K}{2}} \quad (\text{A3})$$

$$\langle T_\tau e^{in\theta(x,\tau)} e^{-in\theta(0,0)} \rangle = \left(\frac{x^2 + (u\tau)^2}{\alpha^2} \right)^{-\frac{n^2}{2K}} \quad (\text{A4})$$

Thus K controls the power law decay of the correlation functions i. e. the scaling dimensions of the operators. More precisely (A3),(A4) imply that $e^{in\phi}$ has dimension $\frac{n^2 K}{2}$ and that $e^{in\theta}$ has dimension $\frac{n^2}{2K}$. u can be interpreted as the velocity of the excitations. To compute the spin-spin correlation functions, one uses the expressions (4).

For $\Delta \neq 0$, correlation functions cannot be obtained exactly anymore. However, the sine-Gordon Hamiltonian can still be studied using renormalization group (RG) techniques^{4,54}. The flow equations for K and Δ are of the Kosterlitz-Thouless form^{55,56}. From (A3), Δ has scaling dimension $2 - \beta^2 K/4$. A small Δ is thus irrelevant for $K > K_c = 8/\beta^2$.

When Δ is irrelevant, K flows to a fixed point value K^* and correlation functions keep their power law character up to logarithms⁵⁴ with the bare value of K replaced by K^* . On the other hand if Δ is relevant, ϕ acquires an expectation value that minimizes the ground state energy and a gap is formed. It can then be shown⁴ that there $\langle f(\phi) \rangle \sim f(\langle \phi \rangle)$ and that $\langle T_\tau e^{i\alpha\theta(x,\tau)} e^{-i\alpha\theta(0,0)} \rangle \sim \exp(-\frac{\sqrt{x^2 + (u\tau)^2}}{\xi})$ where ξ is a correlation length. These results are used extensively in the paper.

APPENDIX B: CALCULATION OF THE VBS ORDER PARAMETER IN THE PHASE IDENTIFIED AS A HALDANE GAP PHASE

In this appendix the identification of sector II of table I with a Haldane gap phase is made more precise. It is a well known fact that a Haldane gap phase has a hidden topological long range order^{45,57}. The order parameter that measures the hidden topological order⁴⁵ is known as the Valence Bond Solid (VBS) order parameter.

The VBS order parameter \mathcal{C} is a nonlocal order parameter defined as

$$\mathcal{C} = \lim_{|i-j| \rightarrow \infty} \langle S_i^z \exp(i\pi \sum_{i < n < j} S_n^z) S_j^z \rangle \quad (\text{B1})$$

In the Haldane gap phase, all the spin-spin correlation functions decay exponentially but $\mathcal{C} \neq 0$. Physically, the fact that the VBS order parameter is non zero indicate that if all the sites with $S^z = 0$ are removed from a spin-1 antiferromagnetic chain

the remaining (“squeezed”) chain has antiferromagnetic order.

In the following, it is shown using bosonization that the mean value of the VBS order parameter is non zero in the phase we identified as a Haldane gap phase. A related problem is the dimerized spin chain⁵⁸, where a bosonization transformation permits to show explicitly the existence of hidden long range order.

In the first place, a bosonized expression of the string operator $\exp(i\pi \sum_{i<n<j} (S_1^z + S_2^z))$ has to be obtained. A naive bosonization would lead to a string operator of the form:

$$\exp(i\sqrt{2}(\phi_s(ja) - \phi_s(ia) + \int_{ia}^{ja} dx e^{i\pi \frac{x}{a}} \frac{\cos(\sqrt{2}\phi_s) \cos(\sqrt{2}\phi_a)}{\pi\alpha})) \quad (\text{B2})$$

That expression has very bad features. First, in the antiferromagnetic phase, the oscillating term is non zero and creates a complicated variable phase, that is extremely tricky to handle because of the unknown lattice renormalizations. In particular it is impossible with that expression to get the mean value of the VBS order parameter. Moreover, in the Haldane phase the mean value of the oscillating term is zero but if it is dropped of the calculation of the VBS order parameter, an unphysical zero VBS order parameter results. However, these incorrect results are only due to an inappropriate boson representation of the string order parameter. The good representation is obtained using the identity: $\exp(i\pi(S_1^z + S_2^z)) = -\exp(i\pi(S_1^z - S_2^z))$. This operator identity comes from the fact that $S_z = \pm 1/2$ so that the string order parameter can now be rewritten

$$\prod_{i<n<j} \exp(i\pi(S_1^z + S_2^z)) = (-)^{j-i-1} \exp(\sum_{i<n<j} i\pi(S_1^z - S_2^z)) \quad (\text{B3})$$

The new form of the string order parameter can be straightforwardly bosonized in the form $(-)^{i-j} e^{i\sqrt{2}(\phi_a(ia) - \phi_a(ja))}$. This one has the correct sign alternation in the antiferromagnetic phase. Using the bosonized expression of $S_1^z + S_2^z = S^z$

$$S^z(x) = \frac{-\sqrt{2}\partial_x \phi_s}{\pi} + e^{i\pi x/a} \frac{2 \cos(\sqrt{2}\phi_s) \cos(\sqrt{2}\phi_a)}{\pi\alpha} \quad (\text{B4})$$

The VBS order parameter is obtained in the form

$$\mathcal{C} = \lim_{|x-y| \rightarrow \infty} \langle \cos(\sqrt{2}\phi_s(x)) \cos(\sqrt{2}\phi_s(y)) \rangle \quad (\text{B5})$$

Since ϕ_s is long range ordered, a non zero VBS order parameter is obtained in the phase identified as a Haldane gap phase in section III A. More precisely, $\mathcal{C} \sim (\langle \cos(\sqrt{2}\phi_s) \rangle)^2$. An alternative derivation can be found in Ref. 18.

APPENDIX C: DERIVATION OF THE BOSONIZED COUPLING FOR A RANDOM Z EXCHANGE

For the sake of simplicity, the derivation is made for one chain.

It is done in two steps. First, one goes to the continuum limit

$$H_{\text{random Jz}} = \int dx J_z(x) S^z(x) S^z(x+a) \quad (\text{C1})$$

with $J_z(x) = aJ_n, x = na, S^z(x) = S_n/a$.

Then, one fermionizes using the Jordan Wigner transformation 2.

After straightforward calculations the $2k_F$ part of $S^z(x)S^z(x+a)$ is obtained in the form

$$- (\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)(x)(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)(x+a) + (\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)(x)(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)(x+a) \quad (C2)$$

Written as above, this is a purely formal expression that contains hidden infinities. For it to make sense, it must be normal ordered. Note that in the preceding cases, only two fermion operators had to be normal ordered. Thus, normal ordering gave only some constants that could be safely dropped from the Hamiltonian. Here, a product of 4 fermion operators has to be normal ordered. Thus, contractions can leave us with generated 2 particle operators that cannot be obtained in a naive bosonization procedure.

Technically, Wick theorem says that any product of operators can be written as a sum of normal ordered products with pairings⁵⁹. The two non zero pairings are the following

$$\begin{aligned} \underline{\psi_R(x)\psi_R^\dagger(x')} &= \frac{1}{2\pi(x-x')} \\ \underline{\psi_L(x)\psi_L^\dagger(x')} &= \frac{-1}{2\pi(x-x')} \end{aligned} \quad (C3)$$

Let us pick up the first term with a non-zero contraction

$$\begin{aligned} \psi_R^\dagger(x)\psi_R(x)\psi_R^\dagger(x+a)\psi_L(x+a) &= : \psi_R^\dagger(x)\psi_R(x)\psi_R^\dagger(x+a)\psi_L(x+a) : \\ &\quad + : \psi_R^\dagger(x)\underline{\psi_R(x)\psi_R^\dagger(x+a)}\psi_L(x+a) : \\ &= : \psi_R^\dagger(x)\psi_R(x)\psi_R^\dagger(x+a)\psi_L(x+a) : + \frac{-1}{2i\pi a} : \psi_R^\dagger(x)\psi_L(x+a) : \end{aligned} \quad (C4)$$

i. e. upon normal ordering, the short distance expansions of operator products generates a 2 fermion term that cannot be obtained doing a naive bosonization.

Looking carefully at expression (C4), one can see that it contains 4 generated 2 fermion terms plus the fully normal ordered 4 fermion terms.

Collecting together the 4 generated 2 fermion terms, one gets

$$\frac{1}{2i\pi a} \left[: \psi_R^\dagger(x)\psi_L(x+a) : - : \psi_L(x)\psi_R(x+a) : + : \psi_R^\dagger(x)\psi_L(x+a) : - : \psi_L^\dagger(x)\psi_R(x+a) : \right] \quad (C5)$$

Upon bosonization, this gives

$$\frac{1}{i\pi a} \times \frac{1}{2\pi a} (e^{i2\phi} - e^{-i2\phi}) = \frac{\sin 2\phi}{(\pi a)^2} \quad (C6)$$

Thus, one obtains

$$H_{\text{random Jz}} = \int dx \frac{J_z^{2k_F}(x)}{(\pi a)^2} \sin 2\phi(x) + \text{less relevant terms} \dots \quad (C7)$$

The normal ordered 4 fermion terms reduces to a term $(\partial_x \phi)^2 \sin 2\phi$. Power counting implies that that term has dimension $2 + K$ and is thus irrelevant irrespective of the value of K .

It is possible to check that (C7) is the correct expression by the following argument: $S_i^z S_{i+1}^z$ is invariant under a rotation around the z axis, so that it cannot depend on θ and since it must have the same scaling dimension at the isotropic point as $S_i^x S_{i+1}^x$, its bosonized form must be either $\cos 2\phi$ or $\sin 2\phi$. Since $S^z \rightarrow -S^z$ corresponds in the bosonization language to $\phi \rightarrow \frac{\pi}{2} - \phi$ the bosonized form of $S_i^z S_{i+1}^z$ must be invariant under such transformation. This rules out the $\cos 2\phi$ and leads to (C7) as the only possible expression for the bosonized form of $S_i^z S_{i+1}^z$. However, it is comforting to be able to derive directly the expression (C7), since it proves that (C7) is not an ad hoc expression that is chosen arbitrarily in order to obtain physically sound results from an inappropriate technique.

REFERENCES

- ¹ Unité Mixte de Recherche du C. N. R. S.
- ² I. Affleck, in *Fields, Strings and Critical Phenomena*, edited by E. Brezin and J. Zinn-Justin (Elsevier Science Publishers, Amsterdam, 1988).
- ³ J. Sólyom, *Adv. Phys.* **28**, 209 (1979).
- ⁴ V. J. Emery, in *Highly Conducting One-Dimensional Solids*, edited by J. T. Devreese and al. (Plenum, New York, 1979), p. 327.
- ⁵ D. A. Tennant, R. A. Cowley, S. E. Nagler, and A. M. Tsvelik, *Phys. Rev. B* **52**, 13368 (1995).
- ⁶ C. Kim *et al.*, *Phys. Rev. Lett.* **77**, 4055 (1996).
- ⁷ P. Jordan and E. Wigner, *Z Phys. B* **47**, 631 (1928).
- ⁸ T. Giamarchi and H. J. Schulz, *Phys. Rev. B* **37**, 325 (1988).
- ⁹ C. Doty and D. S. Fisher, *Phys. Rev. B* **45**, 2167 (1992).
- ¹⁰ K. J. Runge and G. T. Zimanyi, *Phys. Rev. B* **49**, 15212 (1994).
- ¹¹ H. Pang, S. Liang, and J. F. Annett, *Phys. Rev. Lett.* **71**, 4377 (1993).
- ¹² P. Schmitteckert and T. Schulze and C. Schuster and P. Schwab and U. Eckern cond-mat preprint 9706107.
- ¹³ E. Dagotto and T. M. Rice, *Science* **271**, 618 (1996), and references therein.
- ¹⁴ F. D. M. Haldane, *Phys. Rev. Lett.* **50**, 1153 (1983).
- ¹⁵ H. J. Schulz, *Phys. Rev. B* **34**, 6372 (1986).
- ¹⁶ S. P. Strong and A. J. Millis, *Phys. Rev. Lett.* **69**, 2419 (1992).
- ¹⁷ S. Gopalan, T. M. Rice, and M. Sigrist, *Phys. Rev. B* **49**, 8901 (1994).
- ¹⁸ D. G. Shelton, A. A. Nersesyan, and A. M. Tsvelik, *Phys. Rev. B* **53**, 8521 (1996).
- ¹⁹ S. R. White, R. M. Noack, and D. J. Scalapino, *Phys. Rev. Lett.* **73**, 886 (1994).
- ²⁰ A. Sandvik, E. Dagotto, and D. J. Scalapino, *Phys. Rev. B* **53**, 2934 (1996).
- ²¹ K. Hida, *J. Phys. Soc. Jpn.* **60**, 1347 (1991).
- ²² D. Poilblanc, H. Tsunegutsu, and T. M. Rice, *Phys. Rev. B* **50**, 6511 (1994).
- ²³ M. Greven, R. J. Birgeneau, and U. J. Wiese, *Phys. Rev. Lett.* **77**, 1865 (1996).
- ²⁴ M. Takano *et al.*, *Phys. Rev. Lett.* **73**, 3463 (1994).
- ²⁵ B. Chiari, O. Piovesana, T. Tarantelli, and P. F. Zanazzi, *Inorg. Chem.* **29**, 1172 (1990).
- ²⁶ G. Chaboussant *et al.*, *Phys. Rev. B* **55**, 3046 (1997).
- ²⁷ P. R. Hammar and D. H. Reich and C. Broholm and F. Trouw, cond-mat 9708053.
- ²⁸ S. A. Carter *et al.*, *Phys. Rev. Lett.* **77**, 1378 (1996).
- ²⁹ M. Azuma *et al.*, *Phys. Rev. B* **55**, 8658 (1997).
- ³⁰ R. A. Hyman, K. Yang, R. N. Bhatt, and S. M. Girvin, *Phys. Rev. Lett.* **76**, 839 (1997).
- ³¹ R. A. Hyman and K. Yang, *Phys. Rev. Lett.* **78**, 1783 (1997).
- ³² C. Monthus, O. Golinelli and T. Jolicoeur, cond-mat 9705231.
- ³³ M. Steiner, M. Fabrizio and A. O Gogolin, cond-mat preprint 9706096.
- ³⁴ A. O. Gogolin, A. A. Nersesyan, A. M. Tsvelik and L. Yu, cond-mat preprint 9707341.
- ³⁵ E. Orignac and T. Giamarchi, *Phys. Rev. B* **53**, 10453 (1996).
- ³⁶ E. Orignac and T. Giamarchi, *Phys. Rev. B* **56**, 7167 (1997).
- ³⁷ A. Nersesyan, A. Luther, and F. Kusmartsev, *Phys. Lett. A* **176**, 363 (1993).
- ³⁸ H. J. Schulz, *Phys. Rev. B* **53**, 2959 (1996).
- ³⁹ M. P. M. den Nijs, *Phys. Rev. B* **23**, 6111 (1981).

- ⁴⁰ R. Shankar, Int. J. Mod. Phys. B **4**, 2371 (1990).
- ⁴¹ A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).
- ⁴² F. D. M. Haldane, Phys. Rev. Lett. **45**, 1358 (1980).
- ⁴³ A. Luther and Timonen, J. Phys. C **18**, 1439 (1985).
- ⁴⁴ S. P. Strong and A. J. Millis, Phys. Rev. B **50**, 9911 (1994).
- ⁴⁵ M. P. M. den Nijs and K. Rommelse, Phys. Rev. B **40**, 4709 (1989).
- ⁴⁶ E. Orignac and T. Giamarchi, cond-mat preprint, 1997.
- ⁴⁷ W. Apel, J. Phys. C **15**, 1973 (1982).
- ⁴⁸ R. Chithra and T. Giamarchi, Phys. Rev. B **55**, 5816 (1997).
- ⁴⁹ C. Dasgupta and S. K. Ma, Phys. Rev. B **22**, 1305 (1980).
- ⁵⁰ D. S. Fisher, Phys. Rev. B **50**, 3799 (1994).
- ⁵¹ D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).
- ⁵² E. Orignac and T. Giamarchi, cond-mat preprint, 1997.
- ⁵³ S. Fujimoto and N. Kawakami preprint cond-mat 9708097.
- ⁵⁴ T. Giamarchi and H. J. Schulz, Phys. Rev. B **39**, 4620 (1989).
- ⁵⁵ J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974).
- ⁵⁶ J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).
- ⁵⁷ H. Tasaki, Phys. Rev. Lett. **66**, 798 (1991).
- ⁵⁸ K. Hida, Phys. Rev. B **45**, 2207 (1992).
- ⁵⁹ N. N. Bogolyubov and D. V. Shirkov, *Introduction to the theory of quantized fields* (Wiley Interscience, New York, 1957).

TABLES

TABLE I. the 4 sectors of the pure 2 spin chains model with $J_{\perp} < 0$

	I	II	III	IV
K_s	< 1	< 1	> 1	> 1
K_a	$< 1/2$	$> 1/2$	$< 1/2$	$> 1/2$
ϕ_s	$\langle \phi_s \rangle = 0$	$\langle \phi_s \rangle = 0$	massless	massless
θ_a, ϕ_a	$\langle \phi_a \rangle = 0$	$\langle \theta_a \rangle = 0$	$\langle \phi_a \rangle = 0$	$\langle \theta_a \rangle = 0$
phase	Ising AF	Haldane gap	XY2	XY1
Order Parameter	$\cos(\sqrt{2}\phi_s) \cos(\sqrt{2}\phi_a)$?	$e^{i\sqrt{2}\theta_s}$	$e^{i\frac{\theta_s}{\sqrt{2}}} \cos(\frac{\theta_a}{\sqrt{2}})$

TABLE II. the 4 sectors of the pure 2 spin chains model with $J_{\perp} > 0$

	I	II	III	IV
K_s	< 1	< 1	> 1	> 1
K_a	$< 1/2$	$> 1/2$	$< 1/2$	$> 1/2$
ϕ_s	$\langle \phi_s \rangle = \frac{\pi}{\sqrt{8}}$	$\langle \phi_s \rangle = \frac{\pi}{\sqrt{8}}$	massless	massless
θ_a, ϕ_a	$\langle \phi_a \rangle = \frac{\pi}{\sqrt{8}}$	$\langle \theta_a \rangle = \frac{\pi}{\sqrt{2}}$	$\langle \phi_a \rangle = \frac{\pi}{\sqrt{8}}$	$\langle \theta_a \rangle = \frac{\pi}{\sqrt{2}}$
phase	Ising AF	singlet	XY2	XY1
Order Parameter	$\sin(\sqrt{2}\phi_s) \sin(\sqrt{2}\phi_a)$	VBS	$e^{i\sqrt{2}\theta_s}$	$e^{i\frac{\theta_s}{\sqrt{2}}} \sin(\frac{\theta_a}{\sqrt{2}})$

FIGURES

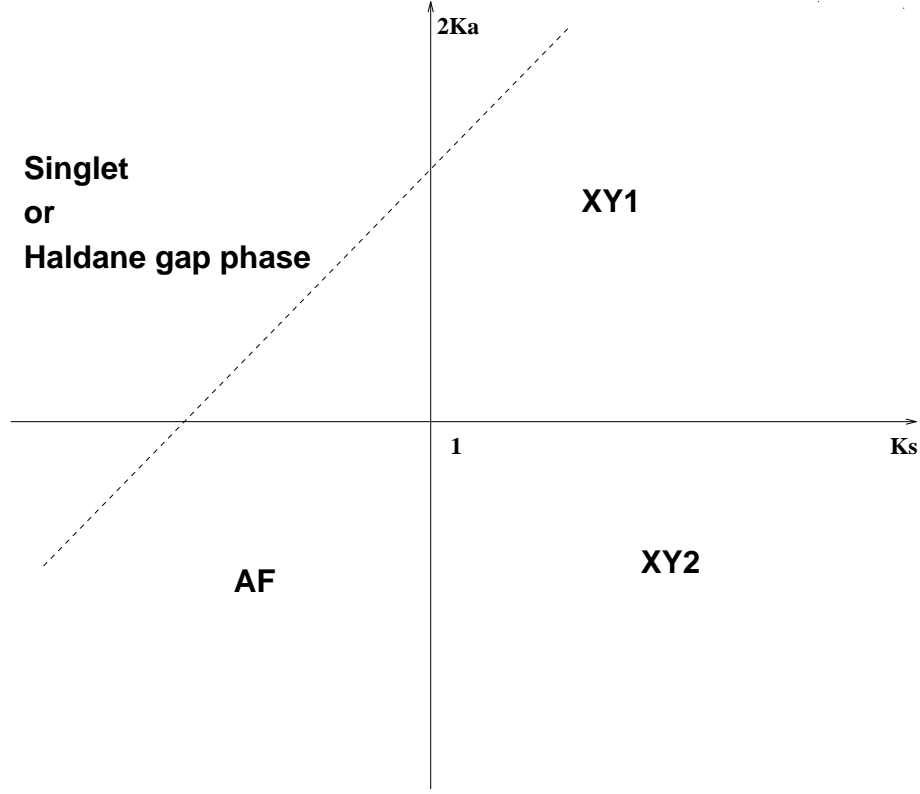


FIG. 1. The phase diagram of the pure 2 chain XXZ model in terms of K_a and K_s . XY1 and XY2 are gapless phases (see text). AF contains antiferromagnetic quasi-long range order. The singlet (antiferromagnetic interchain coupling) or Haldane (ferro. interchain coupling) have a gap to all excitations. The dotted line represent the weak interchain coupling case, when the intrachain anisotropy is varied. The isotropic point is $K_s = K_a = 1/2$. The dashed line corresponds to two coupled XXZ chains.

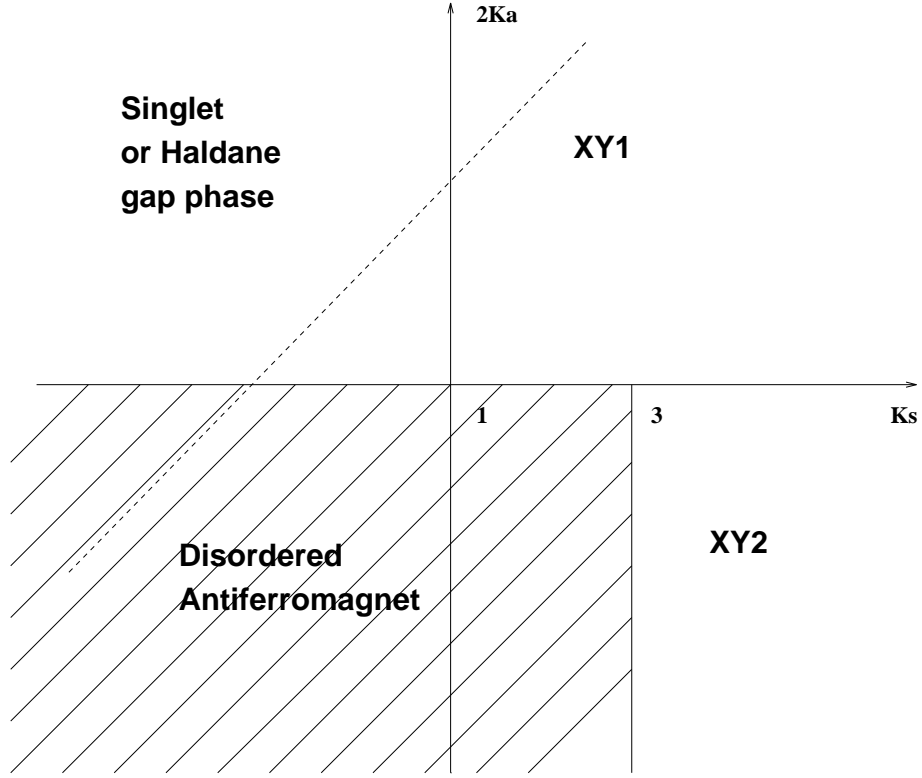


FIG. 2. The phase diagram of the 2 chain XXZ model with a random z field in terms of K_a and K_s . Lines indicate the parts of the phase diagram where disorder is relevant. The singlet phase is stable due to the presence of the gap. Quite surprisingly the massless XY1 phase is now also totally insensitive to weak disorder. The dashed line is for two coupled XXZ chains.

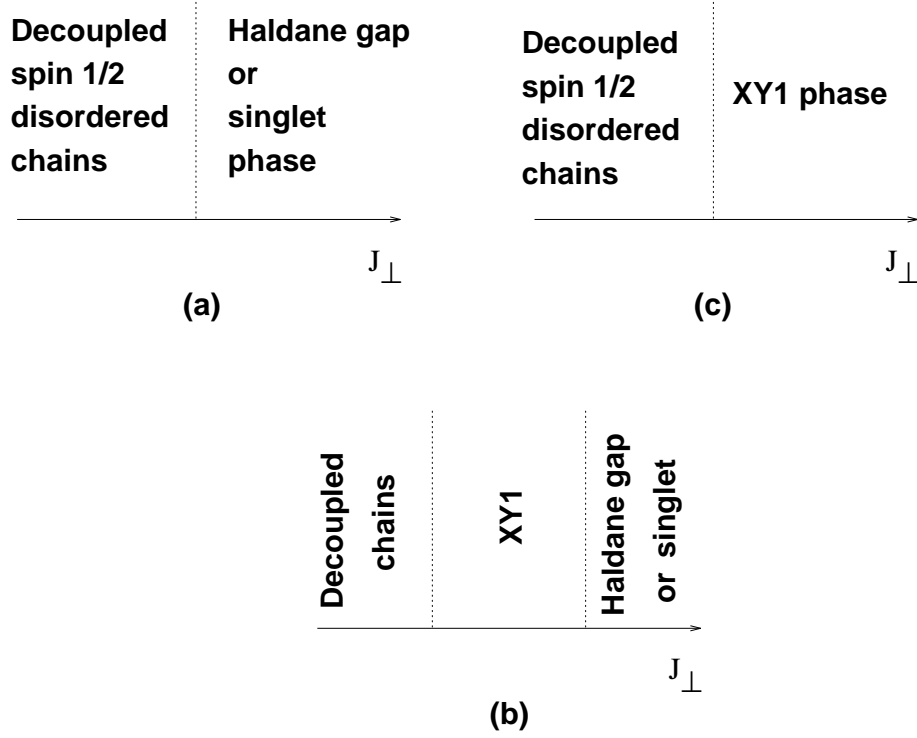


FIG. 3. Phase diagram for fixed disorder as a function of the strength of the interchain coupling J_{\perp} . (a) For $K = 1/2$ the isotropic point. There is a direct transition between decoupled disordered spin 1/2 chains and a stable Haldane or singlet phase. (b) For $K \geq 0.76465$ a disordered XY1 phase exists between the decoupled chains phase and the stable Haldane phase. This phase shows **no** topological order. (c) For $K > 1$ A direct transition between decoupled chains and the stable XY1 phase takes place.

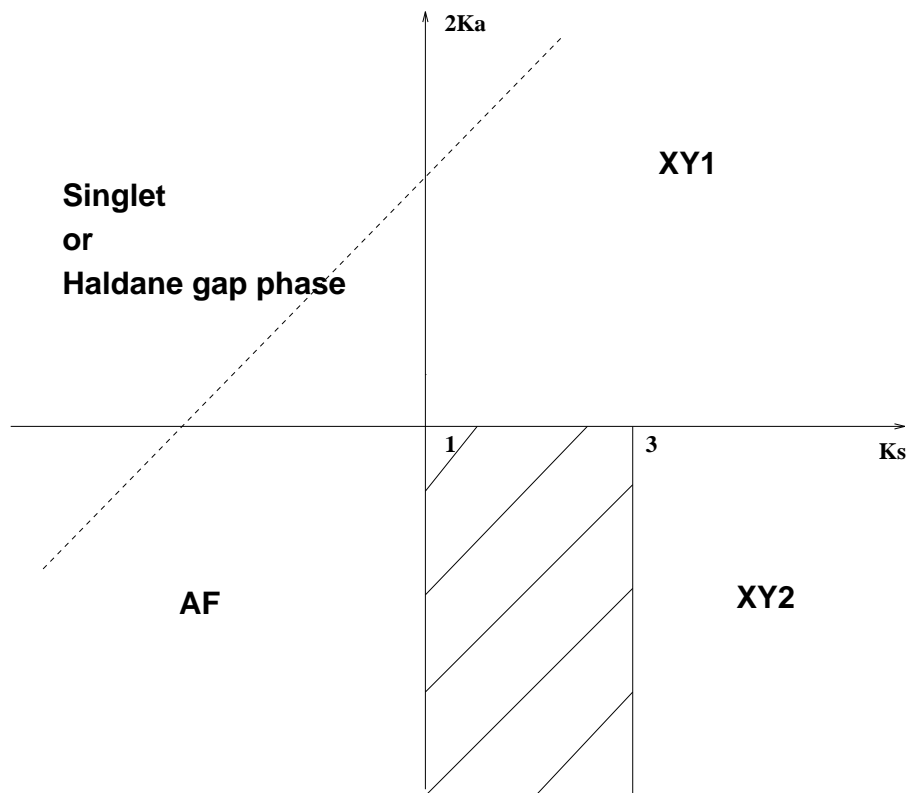


FIG. 4. The phase diagram of the 2 chain XXZ model with a random planar exchange or a random z exchange in terms of K_a and K_s . Lines indicated the parts of the phase diagram where disorder is relevant. Here again the XY1 phase remains unaffected by weak disorder. The dashed line represents coupled XXZ chains.

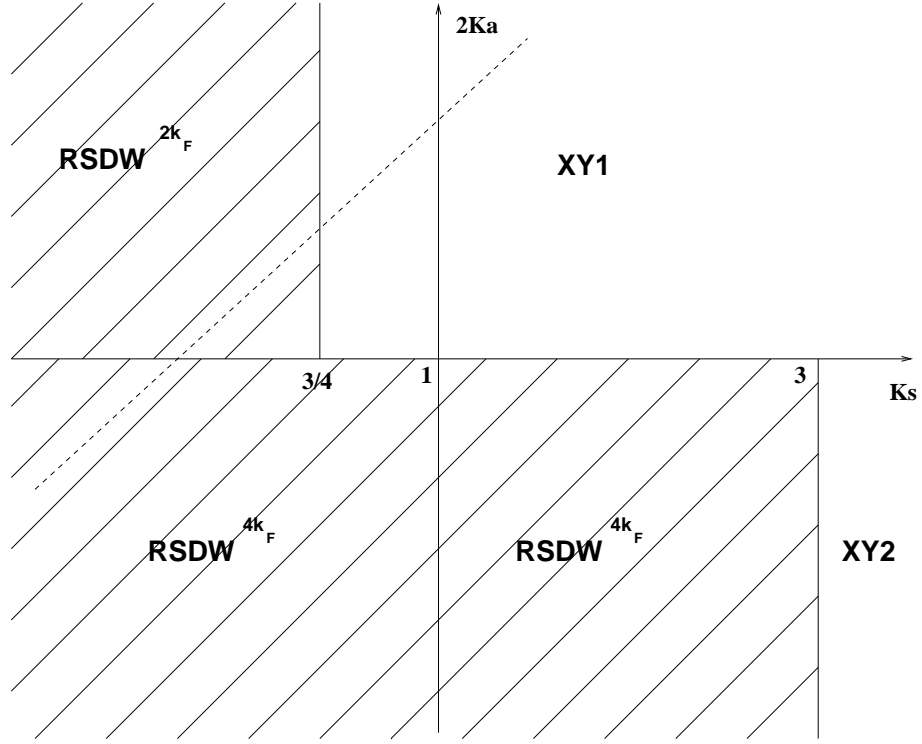


FIG. 5. The phase diagram of the two chain XXZ model with a random z magnetic field under a uniform magnetic field parallel to the z axis. The uniform magnetic field is strong enough to inhibit Haldane gap or singlet gap formation. This results in the apparition of a disordered phase. The dashed line corresponds to coupled XXZ chains.

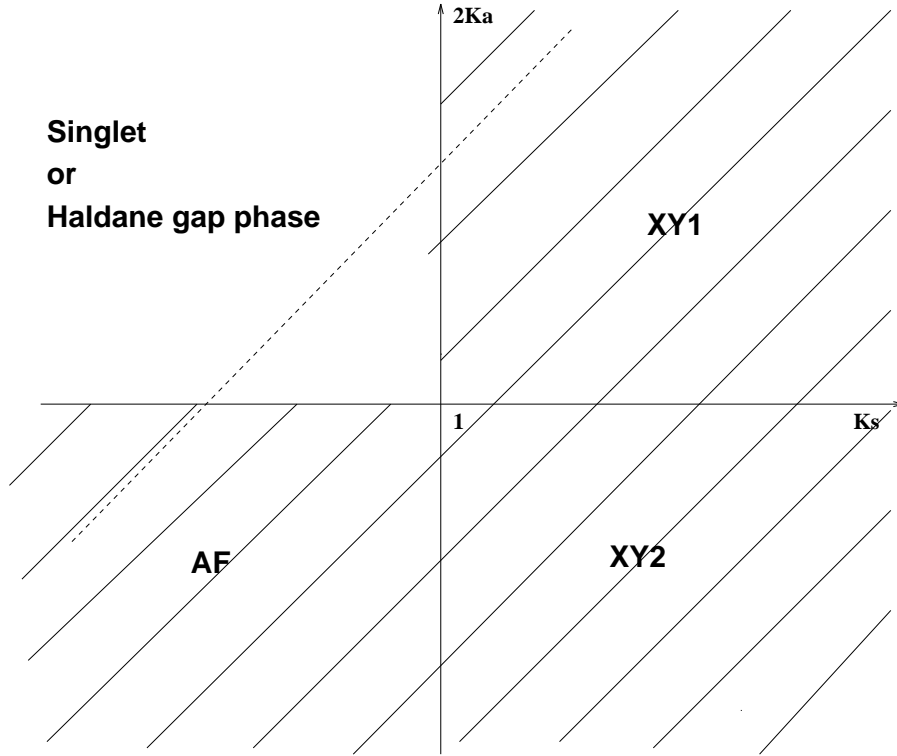


FIG. 6. The phase diagram of the 2 chain XXZ model with a random field in the XY plane or a random planar anisotropy in terms of K_a and K_s . The dashed line corresponds to coupled XXZ chains.

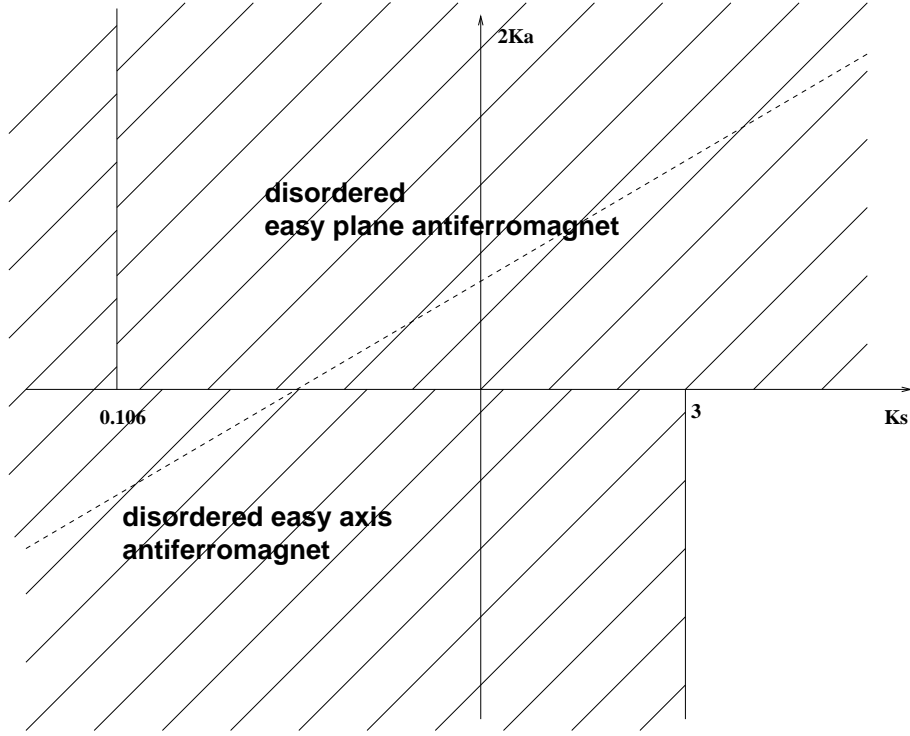


FIG. 7. The phase diagram of the two coupled spin chains in the presence of a random field in the XY plane and a uniform field parallel to the z axis. The dashed line corresponds to two coupled XXZ chains. The uniform field inhibits Haldane gap formation.

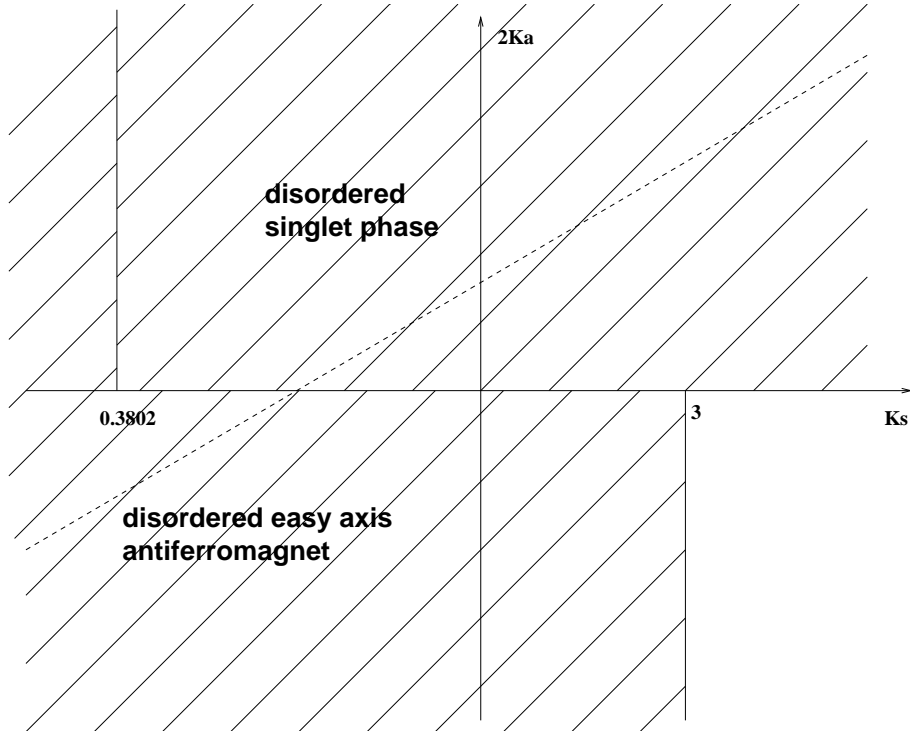


FIG. 8. The phase diagram of the two coupled spin chains in the presence of a random planar anisotropy and a uniform magnetic field parallel to the z axis preventing Haldane gap phase formation. The dashed line corresponds to two XXZ coupled chains.